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The moduli space of curves as a Grothendieck topos?

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Let $\mathcal{M}_{g,n}$ be the moduli spaces of smooth projective algebraic curves of genus g with n marked points. According to

A. Grothendieck

Récoltes et Semailles I, II: Réflexions et témoignage sur un passé de mathématicien. Gallimard, Paris 2023.

these objects are

(Op. cit., p. 194)

les plus beaux, les plus fascinants que j'aie rencontrś en mathématique. Leur seule existence déjà, avec des propriétés à tel point parfaites, m'apparat comme une sorte de miracle (parfaitement bien compris ce qui plus est)... Pour moi, ils renferment en quintessence ce qui est le plus essentiel en géométrie algébrique, savoir la totalité (à peu de choses près) de toutes les courbes algébriques (sur tous les corps de base imaginables)...

Since the seminal paper

P. Deligne and D. Mumford

The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. (1969), 75–109.

the topology and the geometry of $\mathcal{M}_{g,n}$ and of its so-called *Deligne-Mumford compactification* $\overline{\mathcal{M}}_{g,n}$ has been widely investigated, see for instance

E. Arbarello and M. Cornalba

Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Études Sci. Publ. Math. (1998), 97–127.

This research field is still very active, see for instance

J. Bergström, C. Faber, S. Payne

Polynomial point counts and odd cohomology vanishing on moduli spaces of stable curves. Annals of Math. 199 (2024), 1323–1365.

Grothendieck mentions

(Op. cit., p. 237)

des difficultés techniques qui avaient arrêté Mumford, et qui ont été surmontées élégamment dans leur travail par l'introduction des multiplicités modulaires, et d'une "compactification" de celles-ci qui a des propriétés parfaites.

Grothendieck points out

(Op. cit., p. 237)

L'idée même des multiplicités modulaires se trouve, "entre les lignes" tout au moins, dans mes exposés "Teichmüller" au séminaire Cartan, fait à un moment où le langage des sites et des topos n'existait pas encore.

Here the reference is clearly to

A. Grothendieck

Techniques de construction en géométrie analytique. X. Construction de l'espace de Teichmüller. Séminaire Henri Cartan année 13 (1960–61), no. 17 (8 mai 1961).

Next, Grothendieck claims

(Op. cit., p. 237)

Le langage même utilisé par Deligne (”algebraic stack”) là où il y avait tout un langage des sites, topos, multiplicités fait sur mesure pour exprimer ce genre de situation *hides somehow* la provenance de certaines des principales idées mises en oeuvre dans ce travail brillant.

Open Question

Consider $\mathcal{M}_{g,n}$ and its compactification $\overline{\mathcal{M}}_{g,n}$ as Grothendieck toposes. May this (more natural, at least according to Grothendieck) approach have any interesting consequence on our understanding of the geometry of these beautiful spaces?

For moduli spaces of curves, maybe no. But for other moduli spaces...

M. Kontsevich

Enumeration of rational curves via torus actions. The moduli space of curves (Texel Island, 1994), 335–368. Progr. Math., 129. Birkhäuser Boston, Inc., Boston, MA, 1995.

5.1 Higher Genera

We do not know at the moment how to treat higher genus curves in the same way as we treat rational curves. One basic problem is that the moduli space of stable maps is almost never smooth for higher genus. Maybe, this is not a serious obstruction if one adopts the general philosophy of hidden smoothness...

I. Ciocan-Fontanine and M. Kapranov

Derived Hilbert schemes. J. Amer. Math. Soc.15 (2002), 787–815.

(0.1)

The Derived Deformation Theory (DDT) program (see Kontsevich, Ciocan-Fontanine–Kapranov for more details and historical references) seeks to avoid the difficulties related to the singular nature of the moduli spaces in geometry by ”passing to the derived category”, i.e., developing an appropriate version of the (nonabelian) derived functor of the functor of forming the moduli space. The resulting geometric objects are sought to be not ordinary varieties or schemes but rather dg-schemes, i.e., geometric objects whose algebras of functions are commutative differential graded (dg-)algebras and which are considered up to quasi-isomorphism. Moreover, they are expected to be smooth in the sense that the corresponding dg-algebras can be obtained from a smooth commutative algebra in the usual sense by adding projective modules of generators in each negative degree.

I. Ciocan-Fontanine and M. Kapranov

Derived Hilbert schemes. J. Amer. Math. Soc.15 (2002), 787–815.

(5.4.8)

Let Y be a smooth projective variety and let $d \in \mathbb{Z}$. (...) We denote by $\widetilde{M}_{g,n}(Y, d)$ the stack of prestable maps (...). This is an algebraic stack which is nonseparated and possibly nonsmooth. (...) We denote by $\overline{M}_{g,n}(Y, d) \subset \widetilde{M}_{g,n}(Y, d)$ the open substack formed by stable maps. It is known (...) that $\overline{M}_{g,n}(Y, d)$ is a proper Deligne-Mumford stack, in particular, it is separated and of finite type. However, it is not smooth in general, essentially because its tangent spaces (...) are obtained by truncating some naturally arising complex. We now proceed to construct a smooth derived version of $\overline{M}_{g,n}(Y, d)$. (...) Let $\mathcal{S} \subset \widetilde{M}_{g,n}(Y, d)$ be an open substack of finite type. The derived stack of prestable maps of type $\mathcal{S}$ is defined to be $R\widetilde{M}^{\mathcal{S}}_{g,n}(Y, d) =$ (...)

(5.4.8) Theorem. $R\widetilde{M}^{\mathcal{S}}_{g,n}(Y, d)$ is a smooth dg-stack.

J. Lurie

Derived Algebraic Geometry. Ph.D. thesis (2004). Available at https://people.math.harvard.edu/~lurie/papers/DAG.pdf

1.3 Notation and Terminology

We will also make occasional use of the theory of ∞-topoi. This is not entirely necessary: one can reformulate our notion of a derived scheme in a fashion which mentions only ordinary topoi. However, the language of ∞-topoi seems best suited to this purpose.

Definition 4.5.1

A derived scheme is a ringed ∞-topos $(X, \mathcal{O}_X)$ which is locally equivalent to the spectrum of a simplicial commutative ring.