Logic, categories and toposes: from category theory to categorical logic

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Plan

Logic and mathematical reasoning

Logic is the foundation of mathematical reasoning.

And this is a old story:

"*Depuis les Grecs, qui dit mathématique dit démonstration.*" - Nicolas Bourbaki, Éléments de mathématique, in Introduction of Théorie des ensembles

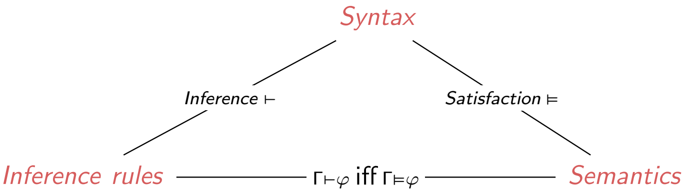
Since Frege and Peano's works and their formalization of arithmetic, a particular logic seemed well suited to formalizing reasoning mathematics: First-order logics, its fragments (equational, geometric, regular, coherent,...), and its higher-order extension.

Since the birth of computer science, a multitude of non-classical logics have emerged: logics whose formulas are not interpreted in Boolean algebra.

- Intuitionistic logic - Heyting algebra (refusal of the law of excluded-middle)
- Modal logic - modal algebra (specification of the qualities of truth)
- Fuzzy logic - residuated lattice (introduction of uncertainty)

Structure of logic

There is a multitude of logics but all of them are characterized by a fundamental distinction between syntax (language, formulas) and semantics (interpretation of the language, truth values of the formulas):



- **Syntax:** gives the rules to build formulas and theories
- **Semantics:** interprets elements of syntax in a mathematical universe
- **Inference:** gives rules to obtain symbolically formulas from other formulas supposed correct.

Categorical logic

Under the leadership of Alfred Tarski, semantics has been formalized to express properties of structures directly in terms of their constituent parts: element of sets, functions between sets, relations on sets, etc.

Now, all these ingredients have an equivalent in category theory:

- elements of sets can be defined by functions $\mathbb{1} \to S$ where S is a set and $\mathbb{1}$ is any singleton;
- functions between sets are morphisms in the category of sets Set
- relations on sets are morphisms $S \to \wp(S)$
- ...

A natural question: may we give a categorical semantics to logics? the answer is yes, and what's more, we have a good candidate, toposes.

Toposes

The notion of topos was first defined by A. Grothendieck at the end of the 1950s to answer problems in algebraic geometry and more precisely to prove Weil's conjectures.

A. Grothendieck's ambition was to abstract the notion of **mathematical space**.

Very quickly, it was observed by A. Grothendieck himself that toposes were a "pastiche" of the sets.

This brought logicians from the 70s under the leadership of Lawvere and his ambition to reformulate the foundations of mathematics using the language of category theory, to define a more general notion, **elementary topos** from a first and complete axiomatization of Grothendieck toposes, Giraud's axioms. Here, the ambition was to abstract the notion of **universe**.

Mathematical logics and categories

What was then observed is that depending on the intended logic, the family of categories used to interpret the formulas is different:

- Cartesian categories interpret Horn logic (formulas are finite conjunctions of atomic formulas);
- Regular categories interpret regular logic (formulas are closed under finite conjunction, and existential quantifications);
- Coherent categories interpret coherent logic (formulas are closed under finite conjunction, finite disjunction, and existential quantifications);
- **Geometric categories interpret geometric logic** (formulas are closed under finite conjunction, infinite disjunction, and the existential quantifier);
- **Heyting categories interpret intuitionistic first-order logic** (formulas are closed under classical propositional connectives, and existential and universal quantifiers);
- **Elementary toposes interpret intuitionistic higher-order logic** (we can quantify on power objects);
- Boolean toposes interpret classical logic.

Syntactic considerations - Signature

Definition 1.

A **signature** Σ is a triple (S, F, R) where:

- S is a set of **sorts**;
- F is a set of **function names** with profile in S^+, i.e. $f : s_1 \times \ldots \times s_n \to s$ (if $n = 0$, then f is called a constant).
- R is a set of **relation names** with profile in S^*, i.e. $r : s_1 \times \ldots \times s_n$.

S^* and S^+ are respectively the set of finite words and the set of unempty finite words on S.

Propositional signature

$\Sigma = (S, F, R)$ where:

- $S = F = \varnothing$
- R is a set of propositional variables.

Let A be a frame (a complete lattice with infinite distributive law). Consider the signature $\Sigma_A = (S_A, F_A, R_A)$ with

- $S_A = F_A = \varnothing$
- $R_A = \{p_a \mid a \in A\}$

Example - Peano's arithmetic

$\Sigma_{Peano} = (S_{Peano}, F_{Peano}, R_{Peano})$ where:

- $S_{Peano} = \{nat\}$
- $F_{Peano} = \left\{ \begin{array}{l} 0 :\to nat \\ succ : nat \to nat \\ + : nat \times nat \to nat \end{array} \right\}$
- $R_{Peano} = \emptyset$.

Example - Presburger's arithmetic

$\Sigma_{Pres} = (S_{Pres}, F_{Pres}, R_{Pres})$ where:

- $S_{Pres} = \{nat\}$
- $F_{Pres} = \left\{ \begin{array}{l} 0 :\rightarrow nat \\ 1 :\rightarrow nat \\ + : nat \times nat \rightarrow nat \end{array} \right\}$
- $R_{Pres} = \{<: nat \times nat\}$

Example - Zermelo-Fraenkel's set theory

$\Sigma_{ZF} = (S_{ZF}, F_{ZF}, R_{ZF})$ where:

- $S_{ZF} = \{Set\}$
- $F_{ZF} = \varnothing$
- $R_{ZF} = \{\in : Set \times Set\}$

Example : Group

$\Sigma_{Grp} = (S_{Grp}, F_{Grp}, R_{Grp})$ where:

- $S_{Grp} = \{G\}$
- $F_{Grp} = \left\{ \begin{array}{l} e :\rightarrow G \\ \_ + \_ : G \times G \rightarrow G \\ \_^{-1} : G \rightarrow G \end{array} \right\}$
- $R_{Grp} = \varnothing$

Example - Stack data type

$\Sigma_{Stack} = (S_{Stack}, F_{Stack}, R_{Stack})$ where:

- $S_{Stack} = \{Stack, Elem\}$
- $F_{Stack} = \left\{ \begin{array}{l} empty :\to Stack \\ push : Elem \times Stack \to Stack \\ pop : Stack \to Stack \\ top : Stack \to Elem \end{array} \right\}$
- $R_{Stack} = \varnothing$

Exampe : Functors on a category

Lat $\mathcal{C}$ be a small category.

$\Sigma_{\mathcal{C}} = (S_{\mathcal{C}}, F_{\mathcal{C}}, R_{\mathcal{C}})$ where:

- $S_{\mathcal{C}} = \{\lceil X \rceil \mid X \in |\mathcal{C}|\}$
- $F_{\mathcal{C}} = \{\lceil f \rceil : \lceil X \rceil \to \lceil Y \rceil \mid f : X \to Y \in \mathcal{C}\}$
- $R_{\mathcal{C}} = \varnothing$

Example: small categories

$\Sigma_{Cat} = (S_{Cat}, F_{Cat}, R_{Cat})$ where:

- $S_{Cat} = \{O, M\}$
- $F_{Cat} = \left\{ \begin{array}{l} 1 : O \to M \\ dom, cod : M \to O \end{array} \right\}$
- $R_{Cat} = \{Comp : M \times M \times M\}$

Syntactic considerations - Terms

Definition 2 (Terms).

Let $\Sigma = (S, F, R)$ be a signature. Let $V = (V_s)_{s \in S}$ be a set of typed variables. The set $T_\Sigma(V) = (T_\Sigma(V)_s)_{s \in S}$ of **Σ-terms with variables in** V is inductively defined as follows: for every $s \in S$, $T_\Sigma(V)_s$ contains

- all variables of V_s and all constant symbols $f :\to s \in F$;
- terms of the form $f(t_1, \ldots, t_n)$ when $f : s_1 \times \ldots \times s_n \to s \in F$, and $(t_1, \ldots, t_n) \in T_\Sigma(V)_{s_1} \times \ldots \times T_\Sigma(V)_{s_n}$.

In the following, we denote $t : s$ to mean that $t \in T_\Sigma(V)_s$.

Syntactic considerations - Atomic formulas

Definition 3 (Atomic formulas)**.**

Let $\Sigma = (S, F, R)$ be a signature and let $V = (V_s)_{s \in S}$ be a set of variables. An **atomic formula** is a sentence of the form

- $t = t'$ where t et t' are terms of the same sort;
- $r(t_1, \ldots, t_n)$ where $r : s_1 \times \ldots \times s_n \in R$ and $(t_1, \ldots, t_n) \in T_\Sigma(V)_{s_1} \times \ldots \times T_\Sigma(V)_{s_n}$.

Syntactic considerations - Formulas

Definition 4 (Formulas)**.**

Let $\Sigma = (S, F, R)$ be a signature and let $V = (V_s)_{s \in S}$ be a set of variables. The set of Σ-**formulas** is inductively defined as follows:

- atomic formulas on Σ and V are Σ-formulas.
- $\bot$ and $\top$ are Σ-formulas.
- if φ and ψ are Σ-formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ are Σ-formulas.
- if $(\varphi_i)_{i \in I}$ is a family of Σ-formulas where I is any index set (not necessarily finite), then $\bigvee_{i \in I} \varphi_i$ is a Σ-formula (geometric logic).
- if φ is a Σ-formula, then $\neg\varphi$ is a Σ-formula.
- φ is a Σ-formula and x is a variable, then $\forall x.\varphi$ and $\exists x.\varphi$ are Σ-formulas.

Syntactic considerations - Sequents and theories

Logical properties in categorical logic are expressed through the notion of **sequents**, that is expressions of the form

$$\varphi \vdash_{\vec{x}} \psi$$

where φ and ψ are formulas of same nature (geometric, coherent, regular, first-order, etc.) and $\vec{x} = (x_1 : s_1, \ldots, x_n : s_n)$ is a **context** i.e. a finite list of variables which contains all variables which occur freely in φ and ψ (i.e. they are not in the scope of a quantifier).

The notion of sequent is needed when dealing with positive and existential logics (geometric, regular, coherent, etc.). In full first-order logic, this notion is not really needed, the statement $\varphi \vdash_{\vec{x}} \psi$ can be replaced by $\top \vdash_{[]} \forall \vec{x}. \varphi \Rightarrow \psi$.

A Σ-**theory** is a set of sequents $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are Σ-formulas.

Example of propositional geometric theory: Frame

Sequents denoting the relations between elements of A

- $p_a \vdash p_b$ when $a \leq b$ $(a, b \in A)$
- $\top \vdash p_1$
- $p_{\bigvee S} \vdash \bigvee_{a \in S} p_a$ $(S \subseteq A)$
- $p_a \wedge p_b \vdash p_{a \wedge b}$ $(a, b \in A)$

Axiomatization of completely prime filters (the finite meets and arbitrary joins in A are treated logically as finite conjunctions and arbitrary disjunctions).

Example of F.O. theories - Peano's arithmetic

Incompleteness and undecidable theory

- 0 is not anyone's successor: $\top \vdash_{[]} \forall x.\neg(succ(x) = 0)$
- any individual other than 0 is someone's successor:

$$\top \vdash_{[]} \forall x.\exists y.(\neg(x = 0) \Rightarrow succ(y) = x)$$

- *succ* is an injective mapping: $\forall x.\forall y.(succ(x) = succ(y) \Rightarrow x = y)$
- recursive definition of +:
  - $\top \vdash_{x} (x + 0 = x)$
  - $\top \vdash_{x,y} (x + succ(y) = succ(x + y))$

  Mathematical induction:

  $(\varphi(x/0) \wedge \forall x.(\varphi \Rightarrow \varphi(x/succ(x))) \vdash_{\vec{x}} \forall x.\varphi$

Example of F.O. theory: Presburger's arithmetic

Decidable theory

- 0 is neutral for +: $\top \vdash_n (n + 0 = n)$
- + is associative: $\top \vdash_{n,m,p} n + (m + p) = (n + m) + p$
- < is a strict, total and discrete order:
  - < is strict and total:
    - Anti-reflexivity: $\top \vdash_n \neg(n < n)$
    - transitivity: $\top \vdash_{x,y,z} (x < y \wedge y < z \Rightarrow x < z)$
    - anti-symmetric: $\top \vdash_{x,y} \neg(x < y \wedge y < x)$
    - total: $\top \vdash_{x,y} (x < y \vee y < x \vee x = y)$
  - < is discrete:
    - < is not dense : $\top \vdash_x \exists y.(x < y \wedge \forall z.(x < z \Rightarrow (y < z \vee y = z)))$
    - every element except 0 has a unique predecessor:

$$\top \vdash_{x,y} (y < x \Rightarrow \exists z.\forall w.(z < x \wedge (z < w \Rightarrow x < w \vee x = w)))$$

- any element is smaller than its direct successor: $\top \vdash_n n < n + 1$

Example of F.O. theory: ZF's set theory

Just a sample of the axioms of the theory.

- **Extensionality axiom**

$$(\forall z. z \in x \Leftrightarrow z \in y) \vdash_{x.y} x = y$$

- **Axiom for union**

$$\top \vdash_x \exists y. \forall z. (z \in y \Leftrightarrow \exists w (w \in x \wedge z \in w))$$

  $(y = \bigcup_{w \in x} w)$.

- **Axiom for powerset**

$$\top \vdash_x \exists y. \forall z. (z \in y \Leftrightarrow \forall w (w \in z \Rightarrow w \in x))$$

  $(y = \wp(x))$

Example of algebraic theory: group theory

- $\top \vdash_x x + e = e + x = x$
- $\top \vdash_{x,y,z} x + (y + z) = (x + y) + z$
- $\top \vdash_x x + x^{-1} = x^{-1} + x = e$

Example of algebraic theory: stack theory

- $\top \vdash_{[]} pop(empty) = empty$
- $\top \vdash_{S,e} pop(push(e, S)) = S$
- $\top \vdash_{S,e} top(push(e, S)) = e$

Example of algebraic theory: theory of functors

- $\top \vdash_x \lceil Id_X \rceil(x) = x$ for all object $X \in |\mathcal{C}|$
- $\top \vdash_x (\lceil g \circ f \rceil)(x) = \lceil g \rceil(\lceil f \rceil(x))$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$

Example of regular theory: theory of small categories

- $\top \vdash_f Comp(f,1,f)$ and $\top \vdash_f Comp(1,f,f)$
- $$\left.\begin{array}{c} Comp(f,g,h) \wedge Comp(g,i,j) \\ \wedge \\ Comp(h,i,k) \wedge Comp(f,j,k') \end{array}\right\} \vdash_{f,g,h,i,j,k,k'} k = k'$$
- $dom(f) = cod(g) \vdash_{f,g} \exists h.Comp(f,g,h)$
- $Comp(f,g,h) \wedge Comp(f,g,h') \vdash_{f,g,h,h'} h = h'$

Example of higher-order theory: Induction

$$\forall a.(\forall b. b \leq a \wedge b \in P) \Rightarrow a \in P \vdash_{(P,x)} x \in P$$

Categorical semantics

Formulas will be interpreted as subobjects in a specific category

Let $\mathcal{C}$ be a category. Let $X \in |\mathcal{C}|$ be an object.

$$a : A \rightarrowtail X \leq_X b : B \rightarrowtail X \text{ iff } \exists x : A \to B, a = b \circ x$$

$$\mathrm{Sub}(X) = \{[a]_{\simeq_X} \mid a : A \rightarrowtail X\}$$

where $\simeq_X$ is the equivalence relation induced by $\leq_X$

When $\mathcal{C}$ has pullbacks,

$$\mathrm{Sub} : \left\{ \begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & \mathrm{Pos} \\ X & \longmapsto & \mathrm{Sub}(X) \\ f : X \to X' & \longmapsto & \mathrm{Sub}(f) : \left\{ \begin{array}{ccc} \mathrm{Sub}(X') & \longrightarrow & \mathrm{Sub}(X) \\ [A' \rightarrowtail X']_{\simeq_{X'}} & \longmapsto & [A \rightarrowtail X]_{\simeq_X} \end{array} \right. \end{array} \right.$$

where $A \rightarrowtail X$ is defined by satifying that the following diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

is a pullback.

Structures

The notion of structures in a category is a natural generalization of the usual Tarskian definition of a (set-based) first-order structure. Categories with finite products are enough to interpret signatures.

Definition 5 (Σ-structure)**.**

Let $\mathcal{C}$ be a category with finite products. Let $\Sigma = (S, F, R)$ be a signature. A Σ-**structure** $\mathcal{M}$ in $\mathcal{C}$ is defined by assigning to:

- each $s \in S$, an object $M_s \in |\mathcal{C}|$,
- each $f : s_1 \times \ldots \times s_n \to s \in F$, a morphism $f^{\mathcal{M}} : M_{s_1} \times \ldots \times M_{s_n} \to M_s \in \mathcal{C}$,
- each $r : s_1 \times \ldots \times s_n \in R$, a subobject $r^{\mathcal{M}} \in Sub(M_{s_1} \times \ldots \times M_{s_n})$.

Categories of Σ-structures

Denote $\Sigma\text{-Str}(\mathcal{C})$ the category of Σ-structures in $\mathcal{C}$, whose morphisms are family of morphisms $(h_s : M_s \to N_s)_{s \in S}$ s.t.:

- for all $f : s_1 \times \ldots \times s_n \to s \in F$, the diagram:

$$\begin{array}{ccc} M_{s_1} \times \ldots \times M_{s_n} & \xrightarrow{f^{\mathcal{M}}} & M_s \\ {\scriptstyle h_{s_1} \times \ldots \times h_{s_n}} \downarrow & & \downarrow {\scriptstyle h_s} \\ N_{s_1} \times \ldots \times N_{s_n} & \xrightarrow[f^{\mathcal{N}}]{} & N_s \end{array}$$

  commutes.

- for all $r : s_1 \times \ldots \times s_n \in R$, there exists a morphism $O \to O'$ s.t. the diagram:

$$\begin{array}{ccc} O & \overset{r^{\mathcal{M}}}{\rightarrowtail} & M_{s_1} \times \ldots \times M_{s_n} \\ \downarrow & & \downarrow {\scriptstyle \mu_{s_1} \times \ldots \times \mu_{s_n}} \\ O' & \underset{r^{\mathcal{N}}}{\rightarrowtail} & N_{s_1} \times \ldots \times N_{s_n} \end{array}$$

  commutes.

Somes properties

Proposition 1.

Σ-Str$(\mathcal{C})$ has finite products.

Proposition 2.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite products. Then, F induces a functor $\Sigma\text{-Str}(F) : \Sigma\text{-Str}(\mathcal{C}) \to \Sigma\text{-Str}(\mathcal{D})$.

More generally, Σ-Str is 2-functorial. Indeeed, we can further show that any natural transformation $\alpha : F \Rightarrow G$ with $F, G : \mathcal{C} \to \mathcal{D}$ induces a natural transformation $\Sigma\text{-Str}(\alpha) : \Sigma\text{-Str}(F) \Rightarrow \Sigma\text{-Str}(G)$.

Evaluation of terms

Notation. Let $\vec{x} = (x_1 : s_1, \ldots, x_n : s_n)$ be a context. Let $\mathcal{M} \in \Sigma\text{-Str}(\mathcal{C})$ be a structure. Denote $M_{\vec{x}} = M_{s_1} \times \ldots \times M_{s_n}$.

Definition 6.

Let $\vec{x}.t$ be a term whose variables are among $\vec{x} = (x_1 : s_1, \ldots, x_n : s_n)$ and $t : s$. Let $\mathcal{M}$ be a Σ-structure. We define by structural induction on terms the **evaluation** of $\vec{x}.t$ in $\mathcal{M}$, denoted $[[\vec{x}.t]]_{\mathcal{M}}$, as:

- if t is a variable x_i, then $[[\vec{x}.t]]_{\mathcal{M}}$ is the canonical projection $p_i : M_{s_1} \times \ldots \times M_{s_n} \to M_{s_i}$;
- if t is of the form $f(t'_1, \ldots t'_m)$ with $t'_1 : s'_1, \ldots t'_m : s'_m$, then $[[\vec{x}.t]]_{\mathcal{M}}$ is the composition

$$M_{\vec{x}} \xrightarrow{([[\vec{x}.t'_1]]_{\mathcal{M}}, \ldots, [[\vec{x}.t'_m]]_{\mathcal{M}})} M_{s'_1} \times \ldots \times M_{s'_m} \xrightarrow{f^{\mathcal{M}}} M_s$$

Some properties

Proposition 3 (Substitution)**.**

Let $\vec{y}.t$ be a term with variables among $\vec{y} = (y_1 : s_1, \ldots, y_n : s_n)$, and $t : s$. Let $\vec{t} = (t_1 : s_1, \ldots, t_n : s_n)$ be a list of terms such that each of them has variables among $\vec{x}$. Then, $[\![\vec{x}.t[\vec{t}/\vec{y}]]\!]$ is the composition $M_{\vec{x}} \xrightarrow{([\![\vec{x}.t_1]\!]_{\mathcal{M}}, \ldots, [\![\vec{x}.t_n]\!]_{\mathcal{M}})} M_{\vec{y}} \xrightarrow{[\![\vec{y}.t]\!]_{\mathcal{M}}} M_s$.

Proposition 4 (Naturality)**.**

Let $h : \mathcal{M} \to \mathcal{N}$ be a morphism of Σ-structures in $\mathcal{C}$, and let $\vec{x}.t$ be a term with $t : s$ Then, the following diagram commutes

$$\begin{array}{ccc} M_{\vec{x}} & \xrightarrow{[\![\vec{x}.t]\!]_{\mathcal{M}}} & M_s \\ {\scriptstyle h_{s_1} \times \ldots \times h_{s_n}} \downarrow & & \downarrow {\scriptstyle h_s} \\ N_{\vec{x}} & \xrightarrow{[\![\vec{x}.t]\!]_{\mathcal{N}}} & N_s \end{array}$$

Satisfaction of formulas

Formulas are interpreted as subobjects defined recursively on the structure of formulas. For this purpose, we need for every object X of a category $\mathcal{C}$ to require $\mathrm{Sub}(X)$ to satisfy a a supplementary property in order to give a mathematical meaning to propositional connectives and quantifiers.

At least, $\mathrm{Sub}(X)$ has to be a bounded lattice - interpretation of $\bot$, $\top$, $\wedge$, and $\vee$.

How to interpret $\Rightarrow$ and quantifiers ?

$\Rightarrow$ can be interpreted in Heyting algebra

Definition 7.

A **Heyting algebra** H is a bounded lattice such that for all $x, y \in H$, there exists a greatest element z satisfying $x \wedge z \leq y$. We denote z by $x \to y$ (adjoint).
The pseudo-complement is defined by $\neg x = x \to \bot$ which satisfies: $x \wedge \neg x = \bot$.

Regarding H as a category, meet $\wedge$ is the product, and then $x \to y$ is the exponential y^x (H is CC). Hence, we have that $\_ \wedge X \dashv X \Rightarrow \_$.

Quantifiers

How to interpret quantifiers ?

Quantifiers can be interpreted through the categorical notion of adjoint functors. Illustrate this in the category of sets Set. Let X and Y be two sets. Let $p : X \times Y \to Y$ be the canonical projection. $\mathrm{Sub}(p) : \wp(Y) \to \wp(X \times Y); S \mapsto \{(x, y) \mid y \in S\}$. Define $\forall_p, \exists_p : \wp(X \times Y) \to \wp(Y)$ as:

$$\forall_p(S) = \{y \mid \forall x \in X, (x, y) \in S\}$$

$$\exists_p(S) = \{y \mid \exists x \in X, (x, y) \in S\}$$

We have:

$$\mathrm{Sub}(p)(S) \subseteq T \text{ iff } S \subseteq \forall_p(T)$$

$$\exists_p(S) \subseteq T \text{ iff } S \subseteq \mathrm{Sub}(p)(T)$$

and then $\exists_p \dashv \mathrm{Sub}(p) \dashv \forall_p$

Categorical interpretation of first-order logic

Definition 8 (Heyting categories)**.**

A **Heyting category** $\mathcal{C}$ is a category such that

- it has finite limits;
- for every object $X \in |\mathcal{C}|$, $Sub(X)$ has finite unions preserved by the pullback functors $Sub(f)$;
- for each morphism $f : X \to Y \in \mathcal{C}$, $Sub(f) : Sub(Y) \to Sub(X)$ has a left-adjoint $\exists f$ and a right-adjoint $\forall f$ which commute with base change i.e.

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array} \qquad\Rightarrow\qquad \begin{array}{ccc} \mathrm{Sub}(X') & \xrightarrow{x^*} & \mathrm{Sub}(X) \\ {\scriptstyle \exists f'/\forall f'}\downarrow & & \downarrow{\scriptstyle \exists f/\forall f} \\ \mathrm{Sub}(Y') & \xrightarrow[y^*]{} & \mathrm{Sub}(Y) \end{array}$$

(Beck-Chevalley condition)

In a Heyting category, $\mathrm{Sub}(X)$ is a Heyting algebra. Given $[m : A \rightarrowtail X], [B \rightarrowtail X] \in \mathrm{Sub}(X)$, $[A \to B \rightarrowtail X] = \forall m[A \wedge B \rightarrowtail A]$.

Lawvere's hyperdoctrines

Definition 9.

A **hyperdoctrine** consists of a category $\mathcal{C}$ with finite limits together with a functor $P : \mathcal{C}^{op} \to \mathrm{Heytalg}$ such that for every morphism $f : X \to Y \in \mathcal{C}$, the Heyting map $P(f) : P(Y) \to P(X)$ has a left-adjunct and a right-adjunct

$$\exists f \dashv P(f) \dashv \forall f$$

that satisfy the Beck-Chevalley condition.

Any Heyting category with its functor $\mathrm{Sub} : \mathcal{C}^{op} \to \mathrm{Pos}$ is a hyperdoctrine.

Syntactic hyperdoctrine: category of context

Let $\mathcal{C}tx$ be the category whose

- objects are all α-equivalence classes $[\vec{x}]_\alpha$ of finite sequences $\vec{x} = (x_1 : s_1, \dots, x_n : s_n)$ of distinct variables in V;
- morphisms are
  - context projections $p_{\vec{y}} : \vec{x} \to \vec{y}$ with $\vec{x} = \vec{z}.\vec{y}$, and
  - sequences of terms $\vec{t} : [\vec{x}] \to [\vec{y}]$ with $\vec{y} = (y_1 : s_1, \dots, y_n : s_n)$ and $\vec{x} = (x_1 : s'_1, \dots, x_m : s'_m)$ is a sequence of Σ-terms $(t_1 : s_1, \dots, t_n : s_n)$ such that the variables of each term t_i are in $\{x_1, \dots, x_m\}$.

Composition is given by substitution, and the identity arrows by variables.

Syntactic hyperdoctrine for intuitionistic first-order logic

Given a formula $\vec{x}.\varphi$, denote $[\vec{x}.\varphi]_{\dashv\vdash} = \{\vec{x}.\psi \mid \vec{x}.\varphi \dashv\vdash \vec{x}.\psi\}$. Then, by defining

$$[\vec{x}.\varphi]_{\dashv\vdash} \wedge [\vec{x}.\psi]_{\dashv\vdash} = [\vec{x}.\varphi \wedge \psi]_{\dashv\vdash} \quad [\vec{x}.\varphi]_{\dashv\vdash} \vee [\vec{x}.\psi]_{\dashv\vdash} = [\vec{x}.\varphi \vee \psi]_{\dashv\vdash}$$

$$[\vec{x}.\varphi]_{\dashv\vdash} \to [\vec{x}.\psi]_{\dashv\vdash} = [\vec{x}.\varphi \Rightarrow \psi]_{\dashv\vdash}$$

the tuple $(P([\vec{x}]), \wedge, \vee, \to \bot, \top)$ is a Heyting algebra (with $P([\vec{x}]) = \{[\vec{x}.\varphi]_{\dashv\vdash} \mid \vec{x}.\varphi\ \Sigma\text{-formula}\})$.

$$P : \mathcal{C}tx^{op} \to \mathrm{Heytalg}; [\vec{x}] \mapsto P([\vec{x}]); \vec{t} : \vec{x} \to \vec{y} \mapsto [\vec{y}.\varphi]_{\dashv\vdash} \mapsto [\vec{x}.\varphi(\vec{y}/\vec{t})]_{\dashv\vdash}$$

Given a projection morphism $p_{\vec{x}} : y.\vec{x} \to \vec{x}$, quantifications are defined by:

$$\exists p_{\vec{x}} : [y.\vec{x}.\varphi]_{\dashv\vdash} \mapsto [\vec{x}.\exists y \varphi]_{\dashv\vdash} \quad \forall p_{\vec{x}} : [y.\vec{x}.\varphi]_{\dashv\vdash} \mapsto [\vec{x}.\forall y \varphi]_{\dashv\vdash}$$

By the rules for the quantifiers, $\exists p_{\vec{x}} \dashv P(p_{\vec{x}}) \dashv \forall p_{\vec{x}}$.

Beck-Chevalley condition holds because $\vec{z}.(\forall y.\varphi)[\vec{x}/\vec{t}] = \vec{z}.\forall x.(\varphi[\vec{x}/\vec{t}])$ and $\vec{z}.(\exists y.\varphi)[\vec{x}/\vec{t}] = \vec{z}.\exists x.(\varphi[\vec{x}/\vec{t}])$with $\vec{t} : \vec{z} \to \vec{x}$ and $\vec{x}.\varphi$ is a Σ-formula (substitution respects quantifiers).

Elementary Toposes: special class of Heyting categories.

Elementary toposes are as a "pastiche" of Set

	Set	***Topos***
Initial object	$\varnothing$	$\varnothing$
Terminal object	$\{x\}$	$\mathbb{1}$
Function space	B^A	**cartesian closed**
Usual operations	$\cap, \cup$	**finite limit** and finite colimit
Subobjects	$\subseteq$	*monics*
Subobject classifier	$\{0,1\}$	$\mathbf{\Omega}$(Stratification of truth)
Characteristic functions	$\begin{array}{ccc} Y & \xrightarrow{!} & \{x\} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle true} \\ X & \xrightarrow[\chi_m]{} & \{0,1\} \end{array}$	$\begin{array}{ccc} Y & \xrightarrow{!} & \mathbb{1} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle true} \\ X & \xrightarrow[\chi_m]{} & \Omega \end{array}$
Power object	$\{0,1\}^X = \wp(X)$	$\Omega^X = PX$
Image factorisation	$A \xrightarrow{f} B = A \xrightarrow{s} Im(f) \hookrightarrow B$	$A \xrightarrow{f} B = A \xrightarrow{e} Im(f) \overset{m}{\rightarrowtail} B$

More formally

Definition 10.

An **elementary topos** $\mathcal{C}$ is a category which is:

- finitely complete
- cartesian closed (CC), and
- has a subobject classifier.

Having a subobject classifier means that there is a monomorphism out of the terminal object $true : \mathbb{1} \rightarrowtail \Omega$ such that for every monomorphism $m : Y \rightarrowtail X$ there is a unique morphism $\chi_m : X \to \Omega$ (called the characteristic morphism of m) such that the following diagram is a pullback:

$$\begin{array}{ccc} Y & \xrightarrow{\;!\;} & \mathbb{1} \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle true} \\ X & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

Example: the category of presheaves $\widehat{\mathcal{C}}$

$\widehat{\mathcal{C}}$ has for objects presheaves $F : \mathcal{C}^{op} \to \mathrm{Set}$ and as morphisms natural transformations between them.

Limits in $\widehat{\mathcal{C}}$ are computed pointwise from the computation of limits in Set

By Yoneda's lemma, we have $G^F(C) \simeq \mathrm{Nat}(\mathrm{Hom}(\_, C), G^F)$. But CC requires that $\mathrm{Nat}(\mathrm{Hom}(\_, C), G^F) \simeq \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(\_, C) \times F, G)$. Then, let us set $G^F(C) = \mathrm{Nat}(\mathrm{Hom}(\_, C) \times F, G)$.

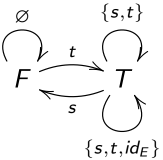
$$\Omega : \begin{cases} \mathcal{C}^{op} & \longrightarrow & \mathrm{Set} \\ A & \longmapsto & \mathrm{Sieve}(A) \\ f : A \to B & \longmapsto & \Omega(f) : \begin{cases} \mathrm{Sieve}(B) & \longrightarrow & \mathrm{Sieve}(A) \\ S & \longmapsto & \{g : C \to A \mid f \circ g \in S\} \end{cases} \end{cases}$$

$true_X : \mathbb{1}(X) \to \Omega(X); \mathbb{1} \mapsto$ maximal sieve on X

Subobject classifier in the category of oriented graphs

Let $\mathcal{C}$ be the category $V \overset{s}{\underset{t}{\rightrightarrows}} E$. Presheaves in $\widehat{\mathcal{C}}$ are then oriented graphs.

Ω is:



Hence, given a subgraph $m : G' \hookrightarrow G$, the characteristic mapping χ_m works as follows:

- all vertices which are not in G' are mapped to F;
- all vertices which are in G' are mapped to T;
- if an edge is not in G', we have 4 possibilities:
  1. edges whose source and target are not in G' are mapped to $\varnothing$;
  2. edges whose source is in G' but the target is not are mapped to s;
  3. edges whose target is in G' but the source is not are mapped to t;
  4. edges whose source and target are in G' are mapped to $\{s, t\}$;
- edges in G' are mapped to $\{s, t, Id_E\}$.

Properties

The following properties hold in any elementary topos $\mathcal{C}$:

- $\mathcal{C}$ is finitely cocomplete, and then it has an intial and a terminal object $\varnothing$ and $\mathbb{1}$, and the unique morphism $\varnothing \to X$ is a monomorphism.
- Every morphism f can be factorized uniquely as $m_f \circ e_f$ where e_f is an epimorphism and m_f is a monomorphism, and then $(A \xrightarrow{f} B) = (A \xrightarrow{e_f} \mathrm{Im}(f) \overset{m_f}{\rightarrowtail} B)$.
- Every object $X \in |\mathcal{C}|$ has a *power object* defined by Ω^X and denoted PX. As a power object, it satisfies the following adjunction property:

$$\mathrm{Hom}_{\mathcal{C}}(X \times Y, \Omega) \simeq \mathrm{Hom}_{\mathcal{C}}(X, PY)$$

$$Sub(X) \simeq Hom_{\mathcal{C}}(X, \Omega) \simeq Hom_{\mathcal{C}}(\mathbb{1}, PX)\ ([m]_{\simeq_X} \mapsto \chi_m)$$

The transpose of Id_{PX} is the characteristic morphism $\in_X : X \times PX \to \Omega$ generalizing the membership notation of set theory.

Algebrization of subobjects

Proposition 5.

Let $\mathcal{C}$ be an elementary topos. Let $X \in |\mathcal{C}|$. $(Sub(X), \leq_X)$ is a bounded lattice.

Given $[f]_\simeq$ and $[g]_\simeq$ in $Sub(X)$ with $f : A \rightarrowtail X$ and $g : B \rightarrowtail X$, their meet is $[A \cap B \rightarrowtail A \overset{f}{\rightarrowtail} X]$ equivalent to $[A \cap B \rightarrowtail B \overset{g}{\rightarrowtail} X]$, and their join is $[A \cup B \rightarrowtail X]$. $(Sub(X), \leq_X)$ is a bounded lattice where $[\varnothing \rightarrowtail X]_\simeq$ and $[id_X]_\simeq$ are the lower and upper bounds.

Theorem 6.

$(Sub(X), \leq)$ is a Heyting algebra.

Elementary toposes are Heyting categories

By Theorem 6, $Sub(X)$ is a Heyting algebra.

Theorem 7.

Let $\mathcal{C}$ be an elementary topos. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. $\mathrm{Sub}(f) : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ has a left-adjunct $\exists f$ and a right-adjunct $\forall f$ which commute with base change.

Let us show how to define quantifiers.

Definition of the existential quantifier

Define the functor

$$\exists : \mathcal{C} \to \mathcal{C}; X \mapsto PX; f \mapsto \exists f : PX \to PY$$

whose the transpose in $Hom_{\mathcal{C}}(PX \times Y, \Omega)$ classifies the image of

$$g : \ni_X \to PX \times X \xrightarrow{Id_{PX} \times f} PX \times Y$$

Using the bijection $\mathrm{Sub}(X) \simeq Hom_{\mathcal{C}}(\mathbb{1}, PX)$ we define

$$\exists f : \begin{cases} Sub(X) & \longrightarrow & Sub(Y) \\ Z \rightarrowtail X & \longmapsto & (\exists f \circ (Z \rightarrowtail X)^{\sharp})^{\sharp} \end{cases}$$

where $(Z \rightarrowtail X)^{\sharp}$ is the transpose of $Z \rightarrowtail X$ in $Hom_{\mathcal{C}}(\mathbb{1}, PX)$.

Definition of universal quantifier

Observe

$$i : Sub(X) \hookrightarrow \mathcal{C}/X$$

Define for $f : Y \to X$

$$f^* : \mathcal{C}/X \to \mathcal{C}/Y ; g : Z \to X \mapsto Z \times_X Y$$

When $f : Y \to \mathbb{1}$, $f^* : \mathcal{C} \to \mathcal{C}/Y ; g : \mathcal{Z} \to \mathbb{1} \mapsto Z \times Y$, and in this case

$$\Pi_{f^*} : \mathcal{C}/Y \to \mathcal{C}/X ; h : Z \to Y \mapsto \mathbb{1} \times_{Y^Y} Z^Y \to \mathbb{1}$$

where

$$\begin{array}{ccc} \mathbb{1} \times_{Y^Y} Z^Y & \longrightarrow & Z^Y \\ \downarrow & & \downarrow{\scriptstyle h^Y} \\ \mathbb{1} & \xrightarrow{Id_Y^{\#}} & Y^Y \end{array}$$

with $Id_Y^{\#}$ transpose of Id_Y from $Hom_{\mathcal{C}}(\mathbb{1} \times Y, Y) \simeq Hom_{\mathcal{C}}(\mathbb{1}, Y^Y)$.

$$f^* \dashv \Pi_{f^*}$$

Let $f : Y \to X$. Observe that $(\mathcal{C}/X)/f \simeq \mathcal{C}/Y$ and then $f^* : \mathcal{C}/X \to \mathcal{C}/Y \simeq (\mathcal{C}/X)/f$ (as previous case). We set $\forall_f = \Pi_{f^*} \circ i$.

Interpretation of F.O. formulas

Let $\mathcal{C}$ be a Heyting category. Let $\mathcal{M} \in \Sigma\text{-Str}(\mathcal{C})$. The **interpretation** of a formula $\vec{x}.\varphi$ in $\mathcal{M}$, denoted $[[\vec{x}.\varphi]]_{\mathcal{M}}$, is recursively defined as a subobject in $\mathrm{Sub}(M_{\vec{x}})$ as follows:

- $[[\vec{x}.r(t_1, \ldots, t_n)]]_{\mathcal{M}} = [O' \rightarrowtail M_{\vec{x}}]$ such that the diagram is a pullback

$$\begin{array}{ccc} O' & \longrightarrow & O \\ \downarrow & & \downarrow r^{\mathcal{M}} \\ M_{\vec{x}} & \xrightarrow{[[\vec{x}.\vec{t}]]_{\mathcal{M}}} & M_{s_1} \times \ldots \times M_{s_n} \end{array}$$

- $[[\vec{x}.t = t']]_{\mathcal{M}}$ is the equalizer of $[[\vec{x}.t]]_{\mathcal{M}}, [[\vec{x}.t']]_{\mathcal{M}} : M_{\vec{x}} \to M_s$.
- $[[\vec{x}.\bot]]_{\mathcal{M}} = [\varnothing \rightarrowtail M_{\vec{x}}]$.
- $[[\vec{x}.\top]]_{\mathcal{M}} = [Id_{M_{\vec{x}}}]$.
- $[[\vec{x}.(\psi \wedge \chi)]]_{\mathcal{M}} = [[\vec{x}.\psi]]_{\mathcal{M}} \wedge [[\vec{x}.\chi]]_{\mathcal{M}}$.
- $[[\vec{x}.(\psi \vee \chi)]]_{\mathcal{M}} = [[\vec{x}.\psi]]_{\mathcal{M}} \vee [[\vec{x}.\chi]]_{\mathcal{M}}$.
- $[[\vec{x}.(\psi \Rightarrow \chi)]]_{\mathcal{M}} = [[\vec{x}.\psi]]_{\mathcal{M}} \to [[\vec{x}.\chi]]_{\mathcal{M}}$.
- $[[\vec{x}.\neg\psi]]_{\mathcal{M}} = [[\vec{x}.\psi]]_{\mathcal{M}} \to [[\vec{x}.\bot]]_{\mathcal{M}}$.
- $[[\vec{x}.\exists y \psi]]_{\mathcal{M}} = \exists p_{\vec{x}.y,\vec{x}}([[\vec{x}.y.\psi]]_{\mathcal{M}})$ where $y : s$ and $p_{\vec{x}.y,\vec{x}} : M_{\vec{x}} \times M_s \to M_{\vec{x}}$ is the canonical projection.
- $[[\vec{x}.\forall y \psi]]_{\mathcal{M}} = \forall p_{\vec{x}.y,\vec{x}}([[\vec{x}.y.\psi]]_{\mathcal{M}})$.

Interpretation of sequents

Let $\mathcal{C}$ be a Heyting category. Let $\mathcal{M} \in \Sigma\text{-Str}(\mathcal{C})$. A sequent $\varphi \vdash_{\vec{x}} \psi$ is **valid** for $\mathcal{M}$ if $[[\vec{x}.\varphi]]_{\mathcal{M}} \leq [[\vec{x}.\psi]]_{\mathcal{M}}$.

A Σ-structure $\mathcal{M}$ is a **model** of a Σ-theory $\mathbb{T}$ if $\mathcal{M}$ valids all sequents in $\mathbb{T}$.

Denote $\mathbb{T}\text{-Mod}(\mathcal{C})$ the full sub-category of $\Sigma\text{-Str}(\mathcal{C})$ whose objects are all Σ-structures which valid all the sequents in $\mathbb{T}$.

Proposition 8.

Let $F : \mathcal{C} \to \mathcal{D}$ be a Heyting functor (i.e. it preserves the Heyting structure of categories and among others the left and right-adjoints $\exists f$ and $\forall f$). Let $\mathcal{M} \in \Sigma\text{-Str}(\mathcal{C})$. Let σ be Σ-sequent. If $\mathcal{M} \vDash \sigma$, then $\Sigma\text{-Str}(F)(\mathcal{M}) \vDash \sigma$. If F is further conservative (it reflects isomorphisms), the opposite implication holds.

And then, $\Sigma\text{-Str}(F) : \mathbb{T}\text{-Mod}(\mathcal{C}) \to \mathbb{T}\text{-Mod}(\mathcal{D})$.

Example: interpretation of the theory of frames

Let B be a frame.

Then, the category of frame morphisms

$$A \to B$$

is identified with

$$\mathbb{T}_A\text{-Mod}(B) \;\; \text{(interpretation of } \mathbb{T}_A \text{ in } B\text{)}$$

A standard model of $\mathbb{T}_A$ can also be described by saying which propositional symbols p_a are assigned the truth value true, and hence by a subset $F \subseteq A$ which is a completely prime filter, i.e.

$$\mathbb{T}_A\text{-Mod}(A) = \text{ completely prime filters of } A$$

Example: interpretation of the theory of functors

Let $\mathcal{E}$ be a Heyting category.

Then, the category of functors

$$\mathcal{C} \to \mathcal{E}$$

is identified with

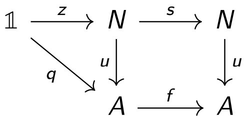
$$\mathbb{T}_{\mathcal{C}}\text{-}\mathsf{Mod}(\mathcal{E})$$

Interpretation of Peano's arithmetic

Peano's arithmetic $\mathbb{T}_P$ is interpretable in all Heyting categories $\mathcal{E}$ with a natural number objet (NNO) N, i.e. an object N together with

- a global elment $z : \mathbb{1} \to N$
- a morphism $s : N \to N$

such that for any $A \in |\mathcal{E}|$, any global element $q : \mathbb{1} \to A$, and any $f : A \to A$, there exists a unique arrow $u : N \to A$ such that



commutes

NNO is initial in $\mathbb{T}_P$-Mod$(\mathcal{E})$.

Interpretation of topological group

Let Top be the category of topological spaces.

This category has all finite limits (cartesian category). It can be used to interpret the group theory $\mathbb{T}_G$

The category of topological groups is identified with

$$\mathbb{T}_G\text{-Mod}(\mathrm{Top})$$

A more standard approach: Kripke-Joyal semantics

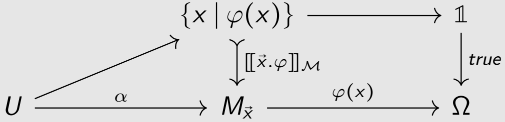
In the Tarskian approach to semantics, the interpretation of formulas is done pointwise. The problem is that this notion of point which could be represented by $\mathbb{1} \to X$ does not necessarily make sense in the theory of topos. A best notion is the one of **generalized element** $U \to X$. This is compatible with the categorical approach which sees objetcts through the morphisms which define them.

Definition 11.

Let $\mathcal{C}$ be an elementary topos. Let $\mathcal{M} \in \Sigma\text{-Str}(\mathcal{C})$. Let $\vec{x}.\varphi$ be a Σ-formula. Let $\alpha : Y \to M_{\vec{x}}$ be a generalized element. Define:

$$\mathcal{M} \vDash_\alpha \vec{x}.\varphi \text{ iff } \alpha \text{ factors through } [[\vec{x}.\varphi]]_{\mathcal{M}}$$

i.e. the following diagram commutes:



Equivalently this means $Im(\alpha) \leq \{x \mid \varphi(x)\}$

Recursive definition on formulas

Theorem 9.

Let $\mathcal{M}$ be a Σ-structure in an elementary topos $\mathcal{C}$.

- $\mathcal{M} \vDash_\alpha \vec{x}.\varphi \wedge \psi$ iff $\mathcal{M} \vDash_\alpha \vec{x}.\varphi$ and $\mathcal{M} \vDash_\alpha \vec{x}.\varphi$;
- $\mathcal{M} \vDash_\alpha \vec{x}.\varphi \vee \psi$ iff there are morphisms $p : V \to Y$ and $q : W \to Y$ such that $p + q : V + W \to Y$ is a jointly epimorphism, and both $\mathcal{M} \vDash_{\alpha \circ p} \vec{x}.\varphi$ and $\mathcal{M} \vDash_{\alpha \circ q} \vec{x}.\psi$;
- $\mathcal{M} \vDash_\alpha \vec{x}.\varphi \Rightarrow \psi$ iff for any morphism $p : V \to Y$, if $\mathcal{M} \vDash_{\alpha \circ p} \vec{x}.\varphi$, then $\mathcal{M} \vDash_{\alpha \circ p} \vec{x}.\psi$;
- $\mathcal{M} \vDash_\alpha \vec{x}.\neg\varphi$ iff for any morphism $p : V \to Y$, if $\mathcal{M} \vDash_{\alpha \circ p} \vec{x}.\varphi$, then $V \simeq \varnothing$;

For the quantifiers, consider a variable $y : s$. Then

- $\mathcal{M} \vDash_\alpha \vec{x}.\exists y.\varphi$ iff there exists an epimorphism $p : V \to Y$ and a generalized element $\beta : V \to M_s$ such that $\mathcal{M} \vDash_{(\alpha \circ p, \beta)} \vec{x}.y.\varphi$;
- $\mathcal{M} \vDash_\alpha \vec{x}.\forall y.\varphi$ iff for every morphism $p : V \to Y$ and every generalized element $\beta : V \to M_s$, one has $\mathcal{M} \vDash_{(\alpha \circ p, \beta)} \vec{x}.y.\varphi$;

Interpretation of higher-order logic

Elementary toposes allow to interpret higher-order logic. Signatures for higher-order logics are also triples $\Sigma = (S, F, R)$ except that the profile of function and relations is inductively defined as follows:

- *basic.* $S \subseteq \Sigma\text{-Typ}$;
- *product.* if $A, B \in \Sigma\text{-Typ}$, then $A \times B \in \Sigma\text{-Typ}$;
- *exponential.* if $A, B \in \Sigma\text{-Typ}$, then $B^A \in \Sigma\text{-Typ}$;
- *power* if $A \in \Sigma\text{-Typ}$, then $PA \in \Sigma\text{-Typ}$.

The family of sets $T_\Sigma(V) = (T_\Sigma(V)_{A \in \Sigma\text{-Typ}}$ with $V = (V_A)_{A \in \Sigma\text{-Typ}}$ is a set of vriables, is then now:

- $V_A \subseteq T_\Sigma(V)_A$, and all constant symbols $f :\to A \in T_\Sigma(V)_A$;
- For each $f : A \to B \in \mathcal{F}$, and each $t \in T_\Sigma(V)_A$, $f(t) \in T_\Sigma(V)_B$.
- For each $t \in T_\Sigma(V)_B$ and $x : A$, then $\lambda x.t \in T_\Sigma(V)_{B^A}$
- For each $t \in T_\Sigma(V)_{B^A}$ and $u \in T_\Sigma(V)_A$, then $t(u) \in T_\Sigma(V)_B$.
- if $t \in T_\Sigma(V)_{A \times B}$, then $fst(t) \in T_\Sigma(V)_A$ and $snd(t) \in T_\Sigma(V)_B$.
- if φ is a Σ-formula and $x \in V_A$, then $\{x \mid \varphi\} \in T_\Sigma(V)_{PA}$

Formulas are defined like for first-order logic to which is added the atomic formula $t \in_A u$ with $t \in T_\Sigma(V)_A$ and $u \in T_\Sigma(V)_{PA}$.

Interpretation of terms

- if t is $fst(u)$ (resp. $snd(u)$) with $u \in T_\Sigma(V)_{A\times B}$, then $[[\vec{x}.t]]_{\mathcal{M}}$ is the composition $\mathcal{M}_{\vec{x}} \xrightarrow{[[\vec{x}.u]]_{\mathcal{M}}} M_A \times M_B \xrightarrow{p_1} M_A$ where $p_1 : M_A \times M_B \to M_A$ (resp. $p_2 : M_A \times M_B \to M_B$)
- $[[\vec{x}.\lambda x.t]]_{\mathcal{M}}$ is the exponential transpose of $[[(\vec{x}.x).t]]_{\mathcal{M}} : M_{\vec{x}} \times M_A \to M_B$.
- $[[\vec{x}.t(u)]]_{\mathcal{M}}$ is

$$M_{\vec{x}} \xrightarrow{([[\vec{x}.t]]_{\mathcal{M}},[[\vec{x}.u]]_{\mathcal{M}})} M_{B^A} \times M_A \xrightarrow{\quad ev \quad} M_B$$

- if t is $\{x \mid \varphi\}$ where $x \in V_A$, then $[[\vec{x}.t]]_{\mathcal{M}}$ is the unique morphism $r : M_{\vec{x}} \to PM_A$ making the diagram a pullback

$$\begin{array}{ccc} R & \longrightarrow & \in_X \\ {\scriptstyle [[(x.\vec{x}).\varphi]]_{\mathcal{M}}}\downarrow & & \downarrow \\ M_A \times M_{\vec{x}} & \xrightarrow[Id_X \times r]{} & M_A \times PM_A \end{array}$$

Interpretation of formulas

$[\![\vec{x}.t \in_A u]\!]_{\mathcal{M}}$ is the subobject $O \rightarrowtail M_{\vec{x}}$ such that the following diagram is a pullback

$$\begin{array}{ccc}
O & \longrightarrow & \in_A \\
{\scriptstyle [\![\vec{x}.t \in_A u]\!]_{\mathcal{M}}} \downarrow & & \downarrow \\
M_{\vec{x}} & \xrightarrow[([\![\vec{x}.t]\!]_{\mathcal{M}},[\![\vec{x}.t']\!]_{\mathcal{M}})]{} & M_A \times PM_A
\end{array}$$

Proposition 8 can be extended to logical functor (i.e. functor which preserves the structure of elementary toposes).

Geometric logic and its interpretation in geometric category

What we observe in practice is that most of the first-order theories naturally arising in Mathematics are geometric.

Problem. How to interpret infinite disjunction ? This leads to the following class of categories

Definition 12.

A category $\mathcal{C}$ is **geometric** if

- it is finitely complete;
- for each $X \in |\mathcal{C}|$,
  - $\mathrm{Sub}(X)$ has arbitrary unions which commute with base change:

$$(X \xrightarrow{f} Y) \Longrightarrow Sub(f)(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} Sub(f)(A_i)$$

.

  - for every $f : X \to Y \in \mathcal{C}$, $Sub(f)$ has a left-adjunct $\exists f$ which commutes with base change.

Grothendieck toposes: a special class of geometric categories

Grothendieck toposes are based on the notion of Grothendieck topologies which generalize categorically the notion of cover in topology of an open by a family of smaller opens.

To present Grothendieck toposes, we first introduce the notions of:

- Sieve;
- Site
- Sheaf on a site.

Sieve

Definition 13.

Lat $\mathcal{C}$ be a category. Let $X \in |\mathcal{C}|$ be an object. A **sieve** on X is a collection of morphisms C with codomain X such that if $f : Y \to X \in C$, then for all $g : Z \to Y \in \mathcal{C}$, $f \circ g : Z \to Y \to X \in C$.

Let $f : Y \to X \in \mathcal{C}$ be a morphism. The **inverse image** by f on a sieve C on X, is the sieve on Y defined by:

$$f^* C = \{g : Z \to Y \mid f \circ g \in C\}$$

Site

Definition 14.

A **Grothendieck topology** on a category $\mathcal{C}$ is a mapping J which assigns to any object $X \in |\mathcal{C}|$ a collection of sieves on X such that:

- *Maximality.* $\bigcup_{Y \in |\mathcal{C}|} Hom_{\mathcal{C}}(Y, X) \in J(X)$.
- *Stability.* For every morphism $f : Y \to X \in \mathcal{C}$, and every sieve $C \in J(X)$, $f^* C \in J(Y)$.
- *Transitivity.* For all sieves C and C' on X such that:
  - $C \in J(X)$ and
  - $f^* C' \in J(Y)$ for every morphism $f : Y \to X \in C$

  $C' \in J(X)$.

Sieves in $J(X)$ are said J-**covering**.
A **site** is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small category and J is a Grothendieck topology on $\mathcal{C}$.

Motivating example of Grothendieck topology

Let T be a topological space. Let us denote by $(\Theta(T), \subseteq)$ its poset of opens. Let $U \in \Theta(T)$ be an open of T. Let $(U_i)_{i \in I}$ be a family of opens such that $\bigcup_{i \in I} U_i = U$. $(U_i)_{i \in I}$ is called an **open covering** of U. Let us define

$$Cov((U_i)_{i \in I}) = \{V \in \Theta(T) \mid \exists i \in I, V \subseteq U_i\}$$

By considering $\Theta(T)$ as a category, we have the site $(\Theta(T), J_{\Theta(T)})$ where $J_{\Theta(T)}$ is the Grothendieck topology

$$J_{\Theta(T)} : U \mapsto \{Cov((U_i)_{i \in I}) \mid (U_i)_{i \in I} \text{ open covering of } U\}$$

Sheaf on a site

Definition 15.

Let $(\mathcal{C}, J)$ be a site. A **sheaf** on $(\mathcal{C}, J)$ is a presehaf $F : \mathcal{C}^{op} \to \mathrm{Set}$ such that for every object $X \in |\mathcal{C}|$ the following amalgamation property is satisfied: for all $C \in J(X)$ and all matching families $S = \{s_f \in F(dom(f)) \mid f : Y \to X \in C\}$ (i.e. for all morphisms $g : Z \to Y \in \mathcal{C}$, $s_{f \circ g} = F(g)(s_f)$), then there exists a single element $s \in F(X)$ such that $s_f = F(f)(s)$ for all $f \in C$.

The category $Sh(\mathcal{C}, J)$ has for objects all sheaves $F : \mathcal{C}^{op} \to \mathrm{Set}$ and for morphisms all natural transformations between them.

A **Grothendieck topos** is any category equivalent to the category of sheaves on a site.

Motivating example of Grothendieck topos

Let T be a topological space. A sheaf $F : \Theta(T)^{op} \to \mathrm{Set}$ is a contravariant functor which satisfies the additional conditions: if $U \subseteq V \in \Theta(T)$, for every $s \in F(V)$, we denote $F(U \subseteq V)(s) = s_{|_U}$

1. if $U \in \Theta(T)$, $(U_i)_{i \in I}$ is an open covering of U, and $s, t \in F(U)$ are elements such that $s_{|_{U_i}} = t_{|_{U_i}}$ for all $i \in I$, then $s = t$;
2. if $U \in \Theta(T)$, $(U_i)_{i \in I}$ is an open covering of U, and given a matching family $\{s_i \in F(U_i) \mid i \in I, (\forall i, j \in I, {s_i}_{|_{U_i \cap U_j}} = {s_j}_{|_{U_i \cap U_j}}\}$, then there is an element $s \in F(U)$ (necessarily unique by (1)) such that $s_{|_{U_i}} = s_i$ for each $i \in I$.

Let us denote $Sh(T)$ the category of sheaves on T. Then, $Sh(\Theta(T), J_{\Theta(T)})$ coincides with $Sh(T)$.

Basic properties

Theorem 10.

Grothendieck toposes are elementary toposes which further satisfy for every $X \in |\mathcal{C}|$ that $\mathrm{Sub}(X)$ is a complete Heyting algebra.

Hence, Grothendieck toposes are geometric categories, and then can be used to interpret the geometric logic.

Inference systems

Modeling of the notion of formal proof. It is a purely syntactic operation which transforms sequents into other sequents independently of the semantic notion of truth.

From axioms, mathematicians allow themselves to deduce theorems by applying a restricted and universally accepted number of forms of reasoning called inference rules. Thus, a system of deduction consists of a choice of a set of axioms and inference rules.

We follow the approach proposed by Gentzen and his sequent calculus which benefits from the recursive structure of formulas.

An inference system consists of a set of inference rules used to infer sequents

$$\frac{\Gamma}{\sigma}$$

σ is infered from a set of sequents Γ.

Axioms

Notation. $\varphi \dashv\vdash \psi$ means both $\varphi \vdash \psi$ et $\psi \vdash \varphi$

- **Identity.** $\varphi \vdash_{\vec{x}} \varphi$
- **Equality.** $\top \vdash_x x = x$
- **Replacement.** $\vec{x} = \vec{y} \wedge \varphi \vdash_{\vec{z}} \varphi[\vec{x}/\vec{y}]$ ($\vec{z}$ contains all the variables of $\vec{x}$ and $\vec{y}$ as well as all the variables of φ)
- **Negation.** $\neg\varphi \dashv\vdash_{\vec{x}} \varphi \Rightarrow \bot$
- **Distributivity.** $\varphi \wedge (\psi \vee \chi) \vdash_{\vec{x}} (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- **Infinite distributivity.** $\varphi \wedge \bigvee_{i \in I} \varphi_i \vdash_{\vec{x}} \bigvee_{i \in I} (\varphi \wedge \varphi_i)$
- **Frobenius axiom.** $\varphi \wedge \exists y.\psi \vdash_{\vec{x}} \exists y.\varphi \wedge \psi$ where $y \notin \vec{x}$.
- **Classical.** $\neg\neg\varphi \vdash_{\vec{x}} \varphi$ ($\mathcal{C}$ is a Boolean topos)
- Higher-order logic
  - **projection.** $\top \vdash_{x.y} fst((x,y)) = x$, $\top \vdash_{x.y} snd((x,y)) = y$ and $\top \vdash_x (fst(x), snd(x)) = x$
  - **application.** $\top \vdash_{\vec{x},z} \lambda y.t(z) = t[y/z]$ and $\top \vdash_u \lambda x.u(y) = u$
  - **membership.** $z \in_A \{y : A \mid \varphi\} \dashv\vdash \varphi[z/y]$ and $\top \vdash_x \{y : A \mid y \in_A x\} = x$ (here $x : PA$)

Structural rules

- *Cut rule*

$$\frac{\varphi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \chi}$$

- *Substitution rule*

$$\frac{\varphi \vdash_{\vec{x}} \psi}{\varphi[\vec{x}/\vec{t}] \vdash_{\vec{y}} \psi[\vec{x}/\vec{t}]}$$

where $\vec{y}$ contains all the variables occuring in $\vec{t}$.

Rules for propositional connectives

- *Conjunction.*

$$\varphi \vdash_{\vec{x}} \top \qquad \varphi \wedge \psi \vdash_{\vec{x}} \varphi \qquad \varphi \wedge \psi \vdash_{\vec{x}} \psi \qquad \varphi \wedge \varphi \dashv\vdash_{\vec{x}} \varphi \qquad \varphi \wedge \psi \dashv\vdash_{\vec{x}} \psi \wedge \varphi$$

$$\frac{\varphi \vdash_{\vec{x}} \psi \qquad \varphi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \psi \wedge \chi}$$

- *Disjunction.*

$$\bot \vdash_{\vec{x}} \varphi \qquad \varphi \vdash_{\vec{x}} \varphi \vee \psi \qquad \psi \vdash_{\vec{x}} \varphi \vee \psi \qquad \varphi \vee \psi \dashv\vdash_{\vec{x}} \psi \vee \varphi$$

$$\frac{\varphi \vdash_{\vec{x}} \chi \quad \psi \vdash_{\vec{x}} \chi}{\varphi \vee \psi \vdash_{\vec{x}} \chi} \qquad \frac{\varphi \vee \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \chi} \qquad \frac{\varphi \vee \psi \vdash_{\vec{x}} \chi}{\psi \vdash_{\vec{x}} \chi}$$

- *Infinite Disjunction.*

$$\frac{\varphi_i \vdash_{\vec{x}} \varphi \text{ for every } i \in I}{\overline{\bigvee_{i \in I} \varphi_i \vdash_{\vec{x}} \varphi}}$$

- *Implication.*

$$\frac{\varphi \wedge \psi \vdash_{\vec{x}} \chi}{\overline{\varphi \vdash_{\vec{x}} \psi \Rightarrow \chi}}$$

Rules for quantifiers

- *Existential quantifier.*

$$\frac{\varphi \vdash_{\vec{x}.y} \psi}{\exists y.\varphi \vdash_{\vec{x}} \psi}$$

  under the condition that y is not free in ψ.

- *Universal quantifier.*

$$\frac{\varphi \vdash_{\vec{x}.y} \psi}{\varphi \vdash_{\vec{x}} \forall y.\psi}$$

Fragments of first-order logic

- Algebraic logic (axioms about equality)
- Horn logic (axioms about equality and finite conjunction)
- Regular logic (axioms about equality, Frobenius axiom, finite conjunction, and existential quantification)
- Coherent logic (axioms about equality, Frobenius axiom, distributivity, finite conjunction, finite disjunction, and existential quantification)
- Geometric logic (axioms about equality, Frobenius axiom, infinite distributivity, finite conjunction, infinite disjunction, and existential quantification)
- Intuitionistic first-order logic (all the finitary rules except the axiom "classical")
- Classical first-order logic (all the finitary rules).

Soundness and completeness

Theorem 11 (Soundness).

Let $\mathbb{T}$ be first-order (resp. geometric) Σ-theory. If a sequent σ is provable from $\mathbb{T}$, then it is valid for all Σ-structures in $\mathbb{T}$-Mod$(\mathcal{C})$ where $\mathcal{C}$ is a Heyting (resp. geometric) category.

The proof is done by structural induction on proof trees.

The method to prove completeness is to define a model of the theory whose validation power is equivalent to the inference. The idea is then to construct a syntactic category whose objects are interpretable in all the models of a theory $\mathbb{T}$.

Method followed

Let $\mathbb{T}$ be a FO (resp. geometric) theory

1. Definition of a covariant functor $\mathbb{T}$-Mod representable by a Heyting (resp. geometric) category $\mathcal{C}_{\mathbb{T}}$, i.e.

   $$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathit{Hom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$$

   and all models of $\mathbb{T}$-Mod$(\mathcal{E})$ can be deduced from a single model (the universal object) $\mathcal{M}_{\mathbb{T}}$.

   When $\mathbb{T}$ is a geometric theory, this result can be restricted to Grothendieck topos. In this case, $\mathbb{T}$-Mod is a contravariant functor, and then $\mathcal{C}_{\mathbb{T}}$ is called a **classifying topos** (Lawvere, Reyes, Makkai, Joyal, Cole, Bénabou,..., from Hakim and Grothendieck's works)

2. Generalisation of Godel's completeness

   $$\text{A sequent } \sigma \text{ is } \mathbb{T}\text{-provable iff it is valid in } \mathcal{M}_{\mathbb{T}}$$

Syntactic category

Definition 16 (α-conversion)**.**

Let $\vec{x}.\varphi$ and $\vec{y}.\psi$ be two Σ-formulas where $\vec{x} = (x_1 : s_1, \ldots, x_n : s_n)$ and $\vec{y} = (y_1 : s_1, \ldots, y_n : s_n)$. $\vec{x}.\varphi$ et $\vec{y}.\psi$ are α-**equivalent** if ψ is obtained from φ by replacing all the free occurences of x_i by y_i. Denote $\{\vec{x}.\varphi\}$ the equivalent class of α-equivalence of $\vec{x}.\varphi$.

Definition 17 (Syntactic category)**.**

Let $\mathbb{T}$ be a Σ-theory. The **syntactic category** $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ has for object all equivalence classes $\{\vec{x}.\varphi\}$, and for morphism $\{\vec{x}.\varphi\} \to \{\vec{y}.\psi\}$ the equivalence class of formulas $[\vec{x}.\vec{y}.\chi]$ for $\dashv\vdash$ which are $\mathbb{T}$-provably functional, i.e. the following sequents are provable in $\mathbb{T}$

$$\varphi \vdash_{\vec{x}} \exists y.\chi \text{ (Existence)}$$
$$\chi \vdash_{\vec{x}.\vec{y}} \varphi \wedge \psi \text{ (Graph of the morphism)}$$
$$\chi \wedge \chi[\vec{z}/\vec{y}] \vdash_{\vec{x}.\vec{y}.\vec{z}} (\vec{y} = \vec{z}) \text{ (Unicity)}$$

Basic properties

Proposition 12.

$\mathcal{C}_{\mathbb{T}}$ is a category.

The identity morphism on $\{\vec{x}.\varphi\}$ is $[\varphi \wedge \vec{x} = \vec{x}']$, and the composition of $[\chi] : \{\vec{x}.\varphi\} \to \{\vec{y}.\psi\}$ and $[\theta] : \{\vec{y}.\psi\} \to \{\vec{z}.\rho\}$ is $[\exists \vec{y}.\chi \wedge \theta]$.

Theorem 13.

If $\mathbb{T}$ is a first-order (resp. geometric) theory, then $\mathcal{C}_{\mathbb{T}}$ is a Heyting (resp. geometric) category. Furthermore, If $\mathbb{T}$ is a higher-order theory, then $\mathcal{C}_{\mathbb{T}}$ is an elementary topos.

Subobjects in syntactic categories

Let $\mathbb{T}$ be a Σ-theory.

Let $\{\vec{x}.\varphi\} \in |\mathcal{C}_{\mathbb{T}}|$.

The **subobjects** of $\{\vec{x}.\varphi\}$ are monomorphisms

$$[\psi \wedge \vec{x} = \vec{x}'] : \{\vec{x}'.\psi[\vec{x}/\vec{x}']\} \rightarrowtail \{\vec{x}.\varphi\}$$

such that the sequent $\psi \vdash_{\vec{x}} \varphi$ is $\mathbb{T}$-provable.

The **order** $\leq$ on $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x}.\varphi\})$ is:

let $[\psi \wedge \vec{x} = \vec{x}'], [\chi \wedge \vec{x} = \vec{x}'] \in \mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x}.\varphi\})$

$$[\psi \wedge \vec{x} = \vec{x}'] \leq [\chi \wedge \vec{x} = \vec{x}']$$
iff
$\psi \vdash_{\vec{x}} \chi$ is $\mathbb{T}$-provable

Universal model

Definition 18.

Let $\mathbb{T}$ be a Σ-theory. The **universal model** $\mathcal{M}_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{C}_{\mathbb{T}}$ is defined as:

- For every $s \in S$, $M_{\mathbb{T}_s} = \{x.\top\}$ where $x : s$.
- for every $f : s_1 \times \ldots \times s_n \to s \in F$, $f^{\mathcal{M}_{\mathbb{T}}} = [f(x_1, \ldots, x_n) = y]$ where $[f(x_1, \ldots, x_n) = y] : \{\vec{x}.\top\} \to \{y.\top\}$, $\vec{x} = (x_1 : s_1, \ldots, x_n : s_n)$ and $y : s$.
- for every $r : s_1 \times \ldots \times s_n \in R$, $[r(x_1, \ldots, x_n)] : \{\vec{x}.r(x_1, \ldots, x_n)\} \rightarrowtail \{\vec{x}.\top\}$

Completeness

Theorem 14.

Let $\mathbb{T}$ be a Σ-theory. Then,

1. The interpretation of $\vec{x}.\varphi$ is the subobject $\{\vec{x}.\varphi\} \rightarrowtail \{\vec{x}.\top\}$ in $\mathrm{Sub}_{\mathcal{C}_{\mathbb{T}}}(\vec{x}.\top)$.
2. A sequent $\varphi \vdash_{\vec{x}} \psi$ is valid in $\mathcal{M}_{\mathbb{T}}$ iff it is $\mathbb{T}$-provable.

1. is proved by structural induction on φ. For 2., if $\varphi \vdash_{\vec{x}} \psi$ is $\mathbb{T}$-provable, then by 1., both are subobjects of $\{\vec{x}.\top\}$, and then $[[\vec{x}.\varphi]]_{\mathcal{M}_{\mathbb{T}}} \leq [[\vec{x}.\psi]]_{\mathcal{M}_{\mathbb{T}}}$. Conversely, if $\varphi \vdash_{\vec{x}} \psi$ is valid in $\mathcal{M}_{\mathbb{T}}$, then $[[\psi]]_{\mathcal{M}_{\mathbb{T}}} \leq [[\varphi]]_{\mathcal{M}_{\mathbb{T}}}$, and then $\psi \vdash_{\vec{x}} \varphi$ is $\mathbb{T}$-provable.

Corollaire 1 (Completeness)**.**

Let $\mathbb{T}$ be a first-order (resp. geometric) theory. All sequents σ which are valid in all models in $\mathbb{T}$-Mod$(\mathcal{C})$ in any Heyting (resp. geometric) category $\mathcal{C}$ are $\mathbb{T}$-provable.

Models as functors

Theorem 15.

Let $\mathbb{T}$ be a first-order (resp. geometric) theory. Then, for every Heyting (resp. geometric) category $\mathcal{D}$, we have the category equivalence $\mathit{Heyt}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \simeq \mathbb{T}\text{-Mod}(\mathcal{D})$ (resp. $\mathit{Geo}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \simeq \mathbb{T}\text{-Mod}(\mathcal{D})$) natural in $\mathcal{D}$ which $F \mapsto \Sigma\text{-Str}(F)(\mathcal{M}_{\mathbb{T}})$.

The reverse equivalence is
$F_{\mathcal{M}} : \{\vec{x}.\varphi\} \mapsto [[\vec{x}.\varphi]]_{\mathcal{M}}; [\theta] : \{\vec{x}.\varphi\} \to \{\vec{y}.\psi\} \mapsto [[(\vec{x}.\vec{y}).\theta]]_{\mathcal{M}}$ (graph of the morphism). Clearly, we have $\Sigma\text{-Str}(F_{\mathcal{M}})(\mathcal{M}_{\mathbb{T}}) \simeq \mathcal{M}$.

$\mathcal{M}_{\mathbb{T}}$ is the model associated to $\mathit{Id}_{\mathcal{C}_{\mathbb{T}}}$.

Example. For Peano's arithmetic, $\mathcal{M}_{\mathbb{T}}$ is an NNO.

Representing object of the functor $\mathbb{T}$-Mod

Denote HeyC the subcategory of Cat whose objects are Heyting categories and morphisms are Heyting functors. We have the functor:

$$\mathbb{T}\text{-Mod} : \begin{cases} \mathrm{HeyC} & \longrightarrow & \mathrm{Cat} \\ \mathcal{C} & \longmapsto & \mathbb{T}\text{-Mod}(\mathcal{C}) \\ F : \mathcal{C} \to \mathcal{D} & \longmapsto & \mathbb{T}\text{-Mod}(F) : \begin{cases} \mathbb{T}\text{-Mod}(\mathcal{C}) & \longrightarrow & \mathbb{T}\text{-Mod}(\mathcal{D}) \\ \mathcal{M} & \longmapsto & \Sigma\text{-Str}(F)(\mathcal{M}) \end{cases} \end{cases}$$

Then, by Theorem 15, $\mathcal{C}_{\mathbb{T}}$ is a representing object of the functor $\mathbb{T}$-Mod. This is also true for the category GeoC and ElemT of geometric categories (with geometric functors) and elementary toposes (with logical functors).

For the category of Grothendieck toposes, $\mathbb{T}\text{-Mod} : \mathcal{BT}\mathbf{op} \to \mathrm{Cat}$ is a contravariant functor. Its representative is called a **classifying topos**.

The category $\mathcal{BT}\mathbf{op}$ of Grothendieck toposes

What notion of morphism ? **Geometric morphism**. Geometric morphisms generalize the case of continuous functions in topological spaces. Let $f : Y \to X$ be a continuous function between two topological spaces. f induces the two functors

$$f^* : Sh(X) \to Sh(Y); F \mapsto [V \subseteq Y \mapsto colim\{F(U) \mid U \subseteq X \wedge V \subseteq f^{-1}(U)\}]$$

$$f_* : Sh(Y) \to Sh(X); F \mapsto [U \subseteq X \mapsto F(f^{-1}(U))]$$

Then, we have that $f^* \dashv f_*$ (and then f^* preserves all small colimits and f_* all small limits). Furthermore, f^* preserves finite limits.

Geometric morphisms and transformations

Definition 19.

Let $\mathcal{C}$ and $\mathcal{D}$ be two Grothendieck toposes. A **geometric morphism** $f : \mathcal{C} \to \mathcal{D}$ consists of a pair of functors $f_* : \mathcal{C} \to \mathcal{D}$ (**direct image** of f) and $f^* : \mathcal{D} \to \mathcal{C}$ (**inverse image** of f) such that $f^* \dashv f_*$ and f^* preserves finite limits.

Definition 20 (Geometric transformation)**.**

A **geometric transformation** $\alpha : f \to g$ between two geometric morphisms $f, g : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\alpha : f^* \Rightarrow g^*$ (equivalently a natural transformation $\alpha : g_* \Rightarrow f_*$).

$\mathcal{BT}$**op** is the 2-category of Grothendieck toposes where 1-morphisms are geometric morphisms and 2-morphisms are geometric transformations. **Geom**$(\mathcal{C}, \mathcal{D})$ is the category whose objects are geometric morphisms and morphisms are geometric transformations.

Example of geometric morphisms: sheafication

Let J be a topology on a small category $\mathcal{C}$. Then,

$$(\widehat{\mathcal{C}} \xrightarrow{j^*} Sh(\mathcal{C}, J), Sh(\mathcal{C}, J) \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}})$$

$j : Sh(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is a geometric morphism where

- $j^* : \widehat{\mathcal{C}} \to Sh(\mathcal{C}, J)$ is the sheafication functor
- j_* is fully faithful (i.e. $j^* \circ j_* \to Id_{Sh(\mathcal{C},J)}$ is an isomorphism), and then $j : Sh(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is an **embedding** of toposes.

Points of topos

The notion of point is *a posteriori* for topos. Let T be a topological space. Any $x \in T$ induces a continuous function $x : \{*\} \to T$, and then a geometric morphism $x : \mathrm{Set} \to Sh(T)$ where

$$x^* : F \mapsto colim\{F(U) \mid U \in \Theta(T) \wedge x \in U\}$$

$$x_* : S \mapsto [U \in \Theta(T) \mapsto (S \text{ if } x \in U; \mathbb{1} \text{ otherwise})$$

Definition 21 (Point).

A **point** in a Grothendieck topos $\mathcal{C}$ is a geometric morphism $x : Set \to \mathcal{C}$.
Denote $\mathbf{Pt}(\mathcal{C}) = \mathbf{Geom}(\mathrm{Set}, \mathcal{C})$.

Diaconescu's equivalence

Theorem 16.

Let $\mathcal{C}$ be a small category with finite limits. Let J be a Grothendeick topology on $\mathcal{C}$. Let $\mathcal{E}$ be a Grothendeick topos. Then,

$$\mathbf{Geom}(\mathcal{E}, Sh(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$ and where $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$ is the category of **flat** and J-continuous functors (i.e. functors which preserve finite limits and transform J-covering families of $\mathcal{C}$ into globally epimorphic families of $\mathcal{E}$).

Sketch of the proof. $f : \mathcal{E} \to Sh(\mathcal{C}, J) \mapsto f^* \circ l : \mathcal{C} \to \mathcal{E}$ where

$l : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} Sh(\mathcal{C}, J)$

$F \mapsto g : \mathcal{E} \to Sh(\mathcal{C}, J)$
such that $g^* = \widehat{F} \circ j_*$ and $g_* = j^* \circ R_F$ where $\widehat{F} : \widehat{\mathcal{C}} \to \mathcal{E}; P \mapsto colim(F \circ \pi_P)$ $(\pi_P : \int P \to \mathcal{C})$ and $R_F : \widehat{\mathcal{C}} \to \mathcal{E}; X \mapsto [Y \mapsto Hom_{\mathcal{E}}(F(Y), X)]$.

Functor $\mathbb{T}$-Mod for Grothendieck toposes

Inverse image functors of geometric morphisms preserve finite limits (by definition) and small colimits (being left-adjunct), so they are geometric functors.

$$\mathbb{T}\text{-Mod} : \begin{cases} \mathcal{BT}\mathbf{op}^{op} & \longrightarrow & \mathrm{Cat} \\ \mathcal{C} & \longmapsto & \mathbb{T}\text{-Mod}(\mathcal{C}) \\ f : \mathcal{C} \to \mathcal{D} & \longmapsto & \mathbb{T}\text{-Mod}(f^*) : \begin{cases} \mathbb{T}\text{-Mod}(\mathcal{D}) & \longrightarrow & \mathbb{T}\text{-Mod}(\mathcal{C}) \\ \mathcal{M} & \longmapsto & \Sigma\text{-Str}(f^*)(\mathcal{M}) \end{cases} \end{cases}$$

Which topology can we associate with $\mathcal{C}_{\mathbb{T}}$ such that $\mathcal{C}_{\mathbb{T}}$ represents this functor?

The syntactic topology $J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$

Definition 22 (Covering family)**.**

Let $\mathbb{T}$ be a geometric theory. Let $\{\vec{y}.\psi\} \in |\mathcal{C}_{\mathbb{T}}|$. A **covering family** of $\{\vec{y}.\psi\}$ is a family of morphisms $([\theta_i] : \{\vec{x}_i.\varphi_i\} \to \{\vec{y}.\psi\})_{i\in I}$ such that the sequent $\psi \vdash_{\vec{y}} \bigvee_{i\in I} \exists \vec{x}_i.\theta_i$ is $\mathbb{T}$-provable (i.e. the union of images is the full object $\{\vec{y}.\psi\}$).

Let us define

$$Cov(([\theta_i]_{i\in I}) = \{[\theta_\chi] : \{\vec{z}.\chi\} \to \{\vec{y}.\psi\} \mid \exists i \in I, \theta_\chi \text{ factors through } \theta_i\}$$

$$J_{\mathbb{T}} : \{\vec{y}.\psi\} \mapsto \{Cov(([\theta_i])_{i\in I}) \mid ([\theta_i])_{i\in I} \text{ is a covering family of } \{\vec{y}.\psi\}\}$$

Syntactic topology for fragments

- Cartesian logic. There exists $i \in I$ such that $Id : \{\vec{y}.\psi\} \to \{\vec{y}.\psi\}$ factors through θ_i
- Regular logic. There exists $i \in I$ such that

$$\psi \vdash_{\vec{y}} \exists \vec{x}_i.\theta_i \text{ is } \mathbb{T}\text{-provable}$$

- Coherent logic. There exists a finite subset $S \subseteq I$ such that

$$\psi \vdash_{\vec{y}} \bigvee_{i \in S} \exists \vec{x}_i.\theta_i \text{ is } \mathbb{T}\text{-provable}$$

Classifying topos

Definition 23.

Let **T** be a geometric theory. A **classifying topos** for $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathcal{C}$, we have the equivalence of categories

$$\mathbf{Geom}(\mathcal{C}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathsf{Mod}(\mathcal{C})$$

natural in $\mathcal{C}$, i.e. for every geometric morphism $f : \mathcal{C} \to \mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathbf{Geom}(\mathcal{D}, \mathrm{Set}[\mathbb{T}]) & \xrightarrow{\simeq} & \mathbb{T}\text{-}\mathsf{Mod}(\mathcal{D}) \\ \big\downarrow {\scriptstyle \_\circ f} & & \big\downarrow {\scriptstyle \mathbb{T}\text{-}\mathsf{Mod}(f^*)} \\ \mathbf{Geom}(\mathcal{C}, \mathrm{Set}[\mathbb{T}]) & \xrightarrow{\simeq} & \mathbb{T}\text{-}\mathsf{Mod}(\mathcal{C}) \end{array}$$

commutes.

$Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is classifying

Theorem 17.

For every geometric theory $\mathbb{T}$, $Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is a classifying topos of $\mathbb{T}$.

This rests on

Lemme 18.

Let $\mathbb{T}$ be a geometric theory. Let $\mathcal{D}$ be a Grothenedieck topos. Then, a functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ is geometric iff it is flat and $J_{\mathbb{T}}$-continuous.

Sketch of the proof of Theorem 17 By Diaconescu's equivalence we have $\mathbf{Geom}(\mathcal{D}, Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})) \simeq Flat_{J_{\mathbb{T}}}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$, and then by the lemma and Theorem 15 we get the expected result.

Every topos is classifying

Theorem 19.

Every Grothendieck topos $\mathcal{E}$ is the classifying topos of some geometric theories.

This rests on

Lemme 20.

Let $\mathcal{C}$ be a small category. Then, a functor $F : \mathcal{C} \to \mathrm{Set}$ is flat iff its category of element $\int F$ is filtered, i.e.:

1. there exists $X \in |\mathcal{C}|$ such that $F(X) \neq \varnothing$.
2. given two elements $x \in F(X)$ and $y \in F(Y)$, there exists an object $Z \in |\mathcal{C}|$, a diagram $X \overset{u}{\leftarrow} Z \overset{v}{\to} Y$ and an element $z \in F(Z)$ such that $F(u)(z) = x$ and $F(v)(z) = y$.
3. given two morphisms $u, v : X \to Y$ in $\mathcal{C}$ and $x \in F(X)$ such that $u(x) = v(x)$, there $w : Z \to X$ and $z \in |\mathcal{Z}|$ such that $u \circ w = v \circ w$ and $w(z) = x$.

Extension of the theory of functors $\mathbb{T}_{\mathcal{C}}$ to the theory of flat and continuous functors

Let $Sh(\mathcal{C}, J)$ be a Grothendeick topos.

- *Functoriality.*
  - $\top \vdash_x f(x) = x$ for all identity morphisms in $\mathcal{C}$
  - $\top \vdash_x f(x) = g(h(x))$ for all morphisms f, g, and h in $\mathcal{C}$ such that $f = g \circ h$
- *Filtering.*
  - $\top \vdash_{[]} \bigvee_{X \in |\mathcal{C}|} \exists x. T$ where $x : X$;
  - $\top \vdash_{x,y} \bigvee_{X \xleftarrow{u} B \xrightarrow{v} Y} \exists z.(u(z) = x \wedge v(z) = y)$ where $z : B$, $x : X$, and $y : Y$;
  - $\top \vdash_x \bigvee_{w:B \to X \in Eq(u,v)} \exists z.w(z) = x$ for all pair of morphisms $u, v : X \to Y$ in $\mathcal{C}$ where $z : B$.
- *Continuity.* $\top \vdash_x \bigvee_{i \in I} \exists y_i.f_i(y_i) = x$ for all J-covering families $(f_i : Y_i \to X)_{i \in I}$ where $y_i : Y_i$ and $x : X$.

For any Grothendieck topos $\mathcal{E}$, $\mathbb{T}_{\mathcal{C}}$-Mod$(\mathcal{E})$ is the category of flat and continuous functors $\mathcal{C} \to \mathcal{E}$.

$Sh(\mathcal{C}, J)$ is the classifying topos of the above theory.

Some consequences

- Taking $\mathcal{D} = \mathrm{Set}$, $\mathbf{Pt}(Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}))$ is equivalent to the category $\mathbb{T}\text{-Mod}(\mathrm{Set})$ of set-theoretic models of $\mathbb{T}$.
- By Theorem 17, we have

$$\mathbf{Geom}(Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}), Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})) \simeq \mathbb{T}\text{-Mod}(\mathcal{C}_{\mathbb{T}})$$

Denote $\mathcal{U}_{\mathbb{T}}$ the model associated to $Id_{Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})}$. For every geometric morphism $f : \mathcal{D} \to Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$, the diagram:

$$\begin{array}{ccc} \mathbf{Geom}(Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}), Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})) & \xrightarrow{\simeq} & \mathbb{T}\text{-Mod}(\mathcal{C}_{\mathbb{T}}) \\ \downarrow {\_ \circ f} & & \downarrow \mathbb{T}\text{-Mod}(f^*) \\ \mathbf{Geom}(\mathcal{D}, Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})) & \xrightarrow{\simeq} & \mathbb{T}\text{-Mod}(\mathcal{D}) \end{array}$$

commutes. So, let $\mathcal{M} \in |\mathbb{T}\text{-Mod}(\mathcal{D})|$. Then, there is a geometric morphism $g_{\mathcal{M}} : \mathcal{D} \to Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ such that $\Sigma\text{-Str}(g_{\mathcal{M}}^*)(\mathcal{U}_{\mathbb{T}}) \simeq \mathcal{M}$ in $\mathcal{D}$, and then the functor

$$f : \mathcal{D} \to Sh(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \mapsto \Sigma\text{-Str}(f^*)(\mathcal{U}_{\mathbb{T}})$$

is an equivalence of categories.

Morita-equivalence for geometric theories

Definition 24.

Two geometric theories $\mathbb{T}$ and $\mathbb{T}'$ are **Morita-equivalent** if they have equivalent classifying toposes, equivalently if there is a natural transformation $\tau : \mathbb{T}\text{-Mod} \Rightarrow \mathbb{T}'\text{-Mod}$ such that for every Grothendieck topos $\mathcal{E}$, $\tau_{\mathcal{E}} : \mathbb{T}\text{-Mod}(\mathcal{E}) \to \mathbb{T}'\text{-Mod}(\mathcal{E})$ is an equivalence of categories.

From this notion, O. Caramello developped the theory of toposes as bridges which consists to find non-trivial connections between properties, concepts and results pertaining to different mathematical theories through the study of the categorical invariants of their classifying toposes (transfer of informations between two theories).

For further reading

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