Local fibrations and Diaconescu's theorem

Léo Bartoli
joint work with Olivia Caramello

Istituto Grothendieck
ETH Zürich

September 2024

Relative toposes

- Relative topos theory allows us to study properties of toposes *relatively to one another*, doing topos theory *over* an arbitrary base topos.
- Some phenomena are more naturally understood by incorporating part of the complexity in the base topos.
- As usual in category theory, the good way of studying objects is by studying how do they interact with one another. In the setting of relative toposes, it is thus important to investigate relative geometric morphisms, and how they can be generated by functors between relative sites.
- The fibrational language and particularly the canonical stack of a relative topos will be the setting allowing us to make a good parallel with the absolute situation, and obtain a good adaptation of notions and results from the ordinary to the relative setting.

From the ordinary to the relative setting: a dictionary

These are the concepts that we are going to translate from the usual topos theory to the relative topos theory:

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	
Categories and functors	
Site $(\mathcal{D}, K)$	
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	
Canonical functor $l : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	

Relative toposes

Definition

A *relative topos*, or a topos over a base topos $\mathcal{F}$ is a geometric morphism $f : \mathcal{E} \to \mathcal{F}$.

- A relative topos comes with its canonical stack: $S_f : \mathcal{F}^{op} \to \mathrm{CAT}$ sending an object F of the base topos $\mathcal{F}$ to the topos $\mathcal{E}/f^*(F)$, with transition morphisms associated to arrows in the base topos $g : F' \to F$ given by pullback:

$$\begin{array}{ccc} E & \xleftarrow{\;h\;} & E' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ f^*(F) & \xleftarrow[f^*(g)]{} & f^*(F') \end{array}$$

- The fibration associated to this stack is $\pi_f : (\mathcal{E} \downarrow f^*) \to \mathcal{F}$.

Relative geometric morphisms

Definition

Let $[f]$ and $[f']$ denote two relative toposes $f : \mathcal{E} \to \mathcal{F}$ and $f' : \mathcal{E}' \to \mathcal{F}$. A relative geometric morphism $g : [f] \to [f']$ is a geometric morphism $g : \mathcal{E} \to \mathcal{E}'$ such that:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad g \quad} & \mathcal{E}' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \mathcal{F} & \end{array}$$

is commutative up to isomorphism.

This says that, for any object F in the basis $g^*(f'^*(F)) \simeq f^*(F)$: the interpretation of the objects coming the basis are fixed by g^*, as if they were parameters.

From the ordinary to the relative setting: a dictionary

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $[f : \mathcal{E} \to \mathcal{F}]$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geom. morph. $g : [f] \to [f']$
Categories and functors	
Site $(\mathcal{D}, K)$	
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	
Canonical functor $l : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	

Fibrations

Recall that an indexed category is a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \text{Cat}$. We can put all the information contained in the functor together into a fibration $p : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

- The category $\mathcal{G}(\mathbb{D})$ has for objects the (x, c) where c is an object of $\mathcal{C}$ and x is an object of $\mathbb{D}(c)$.
- An arrow $(u, f) : (x, c) \to (x', c')$ is the data of an arrow $f : c \to c'$ in the basis, and an arrow $u : x \to \mathbb{D}(f)(x')$ in the category $\mathbb{D}(c)$.
- The arrows of the form $(1, f) : (\mathbb{D}(f)(x), c') \to (x, c)$ are called cartesian arrows. They are, in $\mathcal{G}(\mathbb{D})$, the representatives of the action of the pseudofunctor $\mathbb{D}$ on objects.
- A morphism of fibrations is a functor $A : \mathcal{G}(\mathbb{D}) \to \mathcal{G}(\mathbb{D}')$ preserving cartesian arrows and such that $p'A \simeq p$.

Relative sites

A topos is built from a category with a topology. If we want to do some kind of topos theory inside a base topos, we want to have a notion of category with a topology living in this base topos. The relevant notion is the one of stack, or, more generally, of fibration:

Definition

- For a fibration $p : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ on a site $(\mathcal{C}, J)$, the topology on $\mathcal{G}(\mathbb{D})$ where the covering sieves are those containing a set of cartesian arrows $(1, f_i)_i$ of which the projection $(f_i)_i$ in $(\mathcal{C}, J)$ is covering is called the Giraud's topology.
- We call *relative site* a fibration $p : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$ where K contains Giraud's topology. It induces a relative topos, $C_p : \widehat{\mathcal{G}(\mathbb{D})}_K \to \widehat{\mathcal{C}}_J$, where $C_p^* = a_k(- \circ p)$.

From the ordinary to the relative setting: a dictionary

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $g : \mathcal{E} \to \mathcal{F}$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geom. morph. $g : [f] \to [f']$
Categories and functors	Fibrations and their morphisms
Site $(\mathcal{D}, K)$	Relative site $(\mathcal{G}(\mathbb{D}), K)$
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	
Canonical functor $l : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	

Canonical site for a relative topos

- If we have a relative topos $f : \mathcal{E} \to \mathcal{F}$, we can consider its canonical stack $\pi_f : (\mathcal{E} \downarrow f^*) \to \mathcal{F}$ viewed as a fibration.
- We can endow it with a topology denoted J_f asking for arrows $((u_i, v_i) : (E_i, F_i, \alpha_i : E_i \to f^*(F_i)) \to (E, F, \alpha : E \to f^*(F))_i$ to be covering if and only if their first components $(u_i : E_i \to E)_i$ are jointly epimorphic.
- We have that :

$$\begin{array}{ccc} \widehat{(\mathcal{E} \downarrow f^*)}_{J_f} & \simeq & \mathcal{E} \\ & {\scriptstyle C_{\pi_f}} \searrow \quad \swarrow {\scriptstyle f} & \\ & \mathcal{F} & \end{array}$$

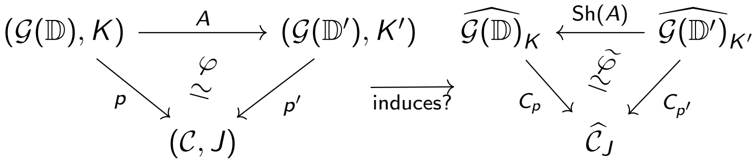
so that every relative topos is induced by at least a relative site: its canonical relative site.

From the ordinary to the relative setting: a dictionary

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $g : \mathcal{E} \to \mathcal{F}$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geom. morph. $g : [f] \to [f']$
Categories and functors	Fibrations and their morphisms
Site $(\mathcal{D}, K)$	Relative site $(\mathcal{G}(\mathbb{D}), K)$
Canonical site $(\mathcal{E}, J_{can}^{\mathcal{E}})$	Canonical rel. site $((\mathcal{E} \downarrow f^*), J_f)$
Canonical functor $l : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	

Relative geometric morphisms

Now that we have relative sites, we want to study which functors will induce relative geometric morphisms:



As the structural morphisms p and p' are comorphism of sites (inducing an inverse image as a precomposition), and we want A to be a morphism of sites (inverse image induced by left Kan extension), it is not trivial to conciliate this two ways of inducing geometric morphisms. Indeed, the natural isomorphism $\varphi : p'A \simeq p$ induces a natural transformation $\tilde{\varphi} : \mathrm{Sh}(A)^* C_p^* \to C_{p'}^*$, which is not necessary an isomorphism.

The η functor

- If we have a comorphism of sites $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ (for example, $(\mathcal{D}, K)$ being a relative site, i.e. $\mathcal{D} = \mathcal{G}(\mathbb{D})$ and K contains its Giraud's topology), we have a dense bimorphism of sites: $\eta_{\mathcal{D}} : (\mathcal{D}, K) \to ((\widehat{\mathcal{D}}_K \downarrow C_p^*), J_{C_p})$ which goes from the site to the canonical relative site of its associated relative topos.
- It is a relative analogue to $l_K : (\mathcal{D}, K) \xrightarrow{y_{\mathcal{D}}} \widehat{\mathcal{D}} \xrightarrow{a_K} \widehat{\mathcal{D}}_K$.
- If we are working with a relative site $p : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$, the η-functor $\eta_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to ((\widehat{\mathcal{G}(\mathbb{D})}_K \downarrow C_p^*), J_{C_p})$ is a morphism of fibrations.
- It acts by sending an object d of $\mathcal{D}$ to the object $\eta_{\mathcal{D}}(d) : l_K(d) \to C_p^*(l_K p(d))$ of $(\widehat{\mathcal{D}}_K \downarrow C_p^*)_{J_{C_p}}$ given as: $a_K(ev_{1_{p(d)}}) : a_K(\mathcal{D}(-, d)) \to a_K(\mathcal{C}(p-, p(d)))$.

Extending A

For "absolute" morphisms of sites, studying if a functor A induces a geometric morphism reduces to see when its extension $\mathrm{Sh}(A)^*$ along the canonical functors is colimits and finite-limits preserving:

$$\begin{array}{ccc} \widehat{\mathcal{D}}_K & \xrightarrow{\mathrm{Sh}(A)^*} & \widehat{\mathcal{D}'}_{K'} \\ l_K \uparrow & & \uparrow l_{K'} \\ (\mathcal{D}, K) & \xrightarrow[A]{} & (\mathcal{D}', K') \end{array}$$

In the relative case, we exhibit the η-extension of a morphism of sites A, which plays the same role:

$$\begin{array}{ccc} ((\widehat{\mathcal{G}(\mathbb{D})}_K \downarrow C_p^*), J_{C_p}) & \xrightarrow{\widetilde{A}} & ((\widehat{\mathcal{G}(\mathbb{D}')}_{K'} \downarrow C_{p'}^*), J_{C_{p'}}) \\ \eta_{\mathbb{D}} \uparrow & & \uparrow \eta_{\mathbb{D}'} \\ (\mathcal{G}(\mathbb{D}), K) & \xrightarrow[A]{} & (\mathcal{G}(\mathbb{D}'), K') \end{array}$$

From the ordinary to the relative setting: a dictionary

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $g : \mathcal{E} \to \mathcal{F}$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geom. morph. $g : [f] \to [f']$
Categories and functors	Fibrations and their morphisms
Site $(\mathcal{D}, K)$	Relative site $(\mathcal{G}(\mathbb{D}), K)$
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	Canonical rel. site $((\mathcal{E} \downarrow f^*), J_f)$
Canonical functor $I : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	Canonical relative functor $\eta : (\mathcal{D}, K) \to ((\widehat{\mathcal{D}}_K \downarrow C_p^*), J_{C_p})$
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	

Definition of the η-extension

If we take a morphism of sites between two relative sites $A : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{G}(\mathbb{D}'), K')$ such that $p'A \simeq p$, we define the *η-extension* of A :

$$\begin{array}{ccc} ((\widehat{\mathcal{G}(\mathbb{D})}_K \downarrow C_p^*), J_{C_p}) & \xrightarrow{\widetilde{A}} & ((\widehat{\mathcal{G}(\mathbb{D}')}_{K'} \downarrow C_{p'}^*), J_{C_{p'}}) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & \widehat{\mathcal{C}}_J & \end{array}$$

$\widetilde{A}$ sends an object $G \xrightarrow{u} C_p^*(F)$ to the object:

$$\mathrm{Sh}(A)^*(G) \xrightarrow{\mathrm{Sh}(A)^*(u)} \mathrm{Sh}(A)^* C_p^*(F) \xrightarrow{\widetilde{\varphi}_F} C_{p'}^*(F)$$

Important theorems

We have a characterization in terms of this extension for A to induce a *relative* geometric morphism. It is the case if and only if, equivalently:

1. $\widetilde{A}$ is a morphism of fibrations
2. $\widetilde{A}$ preserves finite limits *fiberwise*.

Using this characterization, we can obtain:

Theorem

If we have a functor $A : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{G}(\mathbb{D}'), K')$ *such that* $p'A \simeq p$ *which is a morphism of fibrations and of sites, it induces a relative geometric morphism.*

Not a necessary condition

Actually, being a morphism of fibrations is only sufficient for A to induce a relative geometric morphism. Indeed, when we examine the situation, we see that it is only necessary for A to send cartesian arrows on arrows *sent by* $\eta_{\mathbb{D}'}$ to cartesian arrows in the canonical stack of $[C_p]$:

$$\begin{array}{ccc}
((\widehat{\mathcal{G}(\mathbb{D})}_K \downarrow C_p^*), J_{C_p}) & \xrightarrow{\widetilde{A}} & ((\widehat{\mathcal{G}(\mathbb{D}')}_{K'} \downarrow C_{p'}^*), J_{C_{p'}}) \\
\eta_{\mathbb{D}} \uparrow & & \uparrow \eta_{\mathbb{D}'} \\
(\mathcal{G}(\mathbb{D}), K) & \xrightarrow[A]{} & (\mathcal{G}(\mathbb{D}'), K')
\end{array}$$

Locally cartesian arrow

This leads us to the following definition:

Definition

For an arbitrary comorphims $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ we have the η-functor:

$$
\begin{array}{ccc}
(\mathcal{D}, K) & \xrightarrow{\quad \eta_{\mathcal{D}} \quad} & (\widehat{\mathcal{D}}_K \downarrow C_p^*) \\
 & \simeq & \\
 {\scriptstyle p} \searrow & & \swarrow {\scriptstyle \pi_{C_p}} \\
 & (\mathcal{C}, J) &
\end{array}
$$

We say that an arrow $f : d' \to d$ in $\mathcal{D}$ is locally cartesian (with respect to K) if $\eta_{\mathcal{D}}(f)$ is cartesian.

In particular, every cartesian arrow is locally cartesian. Conversely for the canonical relative site of some relative topos, the locally cartesian arrows are exactly the cartesian ones.

Local fibrations

With this notion of locally cartesian arrow comes a notion of local fibration:

Definition

Let $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites. We call it a local fibration if for every arrow $f : c \to p(d)$ in $\mathcal{C}$ there exists a J-covering $(v_i : p(d_i) \to c)_i$ such that $fv_i = p(\widehat{f_i})$ with the $\widehat{f_i} : d_i \to d$ being locally cartesian arrows.

Example

Any fibration is a local fibration: the coverings can be taken as being isomorphisms.

We say that a functor between two local fibrations is a morphism of local fibrations if it preserves locally cartesian arrows and makes the obvious triangle commute.

The central theorem

In this setting, we have the following result:

Theorem

Let $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ *and* $p' : (\mathcal{D}', K') \to (\mathcal{C}, J)$ *be two local fibrations and* $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$ *a morphism of sites such that* $p'A \simeq p$*. We have an equivalence:*

- *The morphism of topos* $\mathrm{Sh}(A)$ *is a morphism of relative toposes, that is:* $C_p\mathrm{Sh}(A) \simeq C_{p'}$*.*
- *The* η*-extension* $\widetilde{A}$ *is a morphism of fibrations.*
- *The functor* A *is a morphism of local fibrations.*

From the ordinary to the relative setting: a dictionary

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $g : \mathcal{E} \to \mathcal{F}$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geom. morph. $g : [f] \to [f']$
Categories and functors	Fibrations and their morphisms
Site $(\mathcal{D}, K)$	Relative site $(\mathcal{G}(\mathbb{D}), K)$
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	Canonical rel. site $((\mathcal{E} \downarrow f^*), J_f)$
Canonical functor $l : (\mathcal{D}, K) \to \widehat{\mathcal{D}}_K$	Canonical relative functor $\eta : (\mathcal{D}, K) \to ((\widehat{\mathcal{D}}_K \downarrow C_p^*), J_{C_p})$
Morphism of sites $A : (\mathcal{D}, K) \to (\mathcal{D}', K')$	Morphisms of sites and of local fibrations

Diaconescu's theorem for local fibrations

From this theorem we obtain the following:

Theorem

Let $(\mathcal{C}, J)$ be a site, $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a local fibration, and $f : \mathcal{E} \to \widehat{\mathcal{C}}_J$ a relative topos with $\pi_f : (\mathcal{E} \downarrow f^* l_J) \to \mathcal{C}$ its canonical stack.
If we denote $\mathrm{LocFibSites}([p], [\pi_f])$ the category of morphisms of local fibrations and sites between $(\mathcal{D}, K)$ and $((\mathcal{E} \downarrow f^* l_J), J_f)$, we have the equivalence:

$$\mathrm{LocFibSites}([p], [\pi_f]) \simeq \mathrm{Topos}/\widehat{\mathcal{C}}_J([f], [C_p])$$

Diaconescu's theorem for fibrations

For fibrations, it appears that the morphisms of local fibrations into the canonical stack are morphisms of fibrations, so:

Theorem

Let $(\mathcal{C}, J)$ *be a site,* $p : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$ *be a relative site, and* $f : \mathcal{E} \to \widehat{\mathcal{C}}_J$ *a relative topos with* $\pi_f : (\mathcal{E} \downarrow f^* l_J) \to \mathcal{C}$ *its canonical stack.*
If we denote $\mathrm{FibSites}([p], [\pi_f])$ *the category of morphisms of fibrations and sites between* $(\mathcal{G}(\mathbb{D}), K)$ *and* $((\mathcal{E} \downarrow f^* l_J), J_f)$*, we have the equivalence:*

$$\mathrm{FibSites}([p], [\pi_f]) \simeq \mathrm{Topos}/\widehat{\mathcal{C}}_J([f], [C_p])$$