Groups, Groupoids, Stacks and Representation Theory.

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The goals of my talk are

(i) to explain that the standard definition (definition R) of a representation of a group is a little bit naïve

(ii) to propose an alternative – categorical – definition (definition M)

(iii) to show that the categorical definition is more "correct" one.

For simplicity I will fix a field k and work with representations in vector spaces over k.

First consider the case of a discrete group G.

Definition (R). A representation of the group G is a pair (V, π), where V is a k-vector space and π is a morphism $\pi : G \to Aut(V)$.

The category of such representations we denote by R(G).

(i)A **groupoid** is a small category $\mathcal{X}$ in which all morphisms are invertible (i.e. isomorphisms).

A groupoid can be considered geometrically as some kind of a space.

(ii)A **sheaf** (of *k*-vector spaces) on a groupoid $\mathcal{X}$ is a contravariant functor F from $\mathcal{X}$ to the category of *k*-vector spaces, $F : \mathcal{X}^0 \rightarrow Vec(k)$.

The category of such sheaves we denote by $Sh(\mathcal{X})$.

Basic groupoid. Starting with a group G we construct its **basic groupoid** $\mathrm{B}G$. Namely, $\mathrm{B}G$ is the category of G-torsors T.

Remind that a G-torsor is a G-set T such that T is not empty and G acts on T simply transitively.

It means that the natural morphism $\gamma : G \times T \to T \times T;\ (g, t) \to (gt, t)$ is an isomorphism.

Morphisms of G-torsors are just morphisms of G-sets.

If Z is a G-set we define a quotient groupoid BG(Z). It is also denoted $G\backslash Z$.

In case when Z is a one point set pt we get the basic groupoid, $\mathrm{B}G(pt) = \mathrm{B}G = G\backslash pt$.

The object of the category BG(Z) is a pair (T, p), where T is a G-torsor and $p : T \to Z$ a morphism of G-sets. Morphisms are just G-morphisms over Z.

Definition (M). Given a (discrete) group G let us define the category $\mathrm{M}(G)$ of stacky G-modules to be the category of sheaves on the basic groupoid $\mathrm{B}G$
$\mathrm{M}(G) := Sh(\mathrm{B}G)$.

Proposition. The category $\mathrm{M}(G)$ is canonically equivalent to the category $\mathrm{R}(G)$.

Category BG is equivaling to the full subcategory $T(G) \subset BG$ generated by the trivial torsor T (the set G with the left action of G).

Hence the category $M(G) = Sh(BG)$ is equivalent to the category $Sh(T(G)) = R(G)$

Similarly, one proves that the category $Sh(G\backslash Z)$ of sheaves on a quotient groupoid is canonically equivalent to the category $Sh_G(Z)$ of G-equivariant sheaves on Z.

9 Two definitions –Categories R(G) and M(G)

Thus we have two equivalent definitions – (R) and (M) – of the category R(G) = M(G).

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In slides 11 – 13 I explain that, nevertheless, even in the case of sets, the definition (M) is better.

The main advantage of the definition (M) is that the category M(G) has **natural generalizations** in a large variety of situations.

I will discuss this in slides 14 – 20

In slides 21-31 I will discuss in detail the key example – Representations of p-adic algebraic groups.

We say that a groupoid X is connected if it is not empty and all its objects are isomorphic.

Two statements:

(i) Any groupoid X has a canonical decomposition as a disjoint union of connected groupoids (they are called the connected components of X).

(ii) Let X be a connected groupoid. Fix an object $x \in \mathrm{X}$ and consider the group H opposite to the group $Mor(x, x)$. Then the category X is canonically equivalent to the category $\mathrm{B}H$ with equivalence described by $y \to Mor(x, y)$.

Suppose we have a group of symmetries Γ that acts on a group G. Then it acts on the category $\mathrm{R}(G)$.

However, in many cases it turns out that we can find a large group of symmetries Γ that acts on the groupoid $\mathrm{B}G$, and hence on the category $\mathrm{M}(G) = \mathrm{R}(\mathrm{G})$, but does not act on the group G.

So, if we use the definition (R), it would be difficult to see these Γ-symmetries.

12 Example – Fundamental groupoid

Let Z be a nice topological space.

Consider the fundamental groupoid $\Pi(Z)$ – its objects are the points of Z, morphisms $p \in Mor(z, w)$ are homotopy classes of paths from z to w, and composition is given by concatenation.

If the space Z is connected, then the groupoid $\Pi(Z)$ is connected.

If z is a point of Z, then the corresponding group G=$Aut(z)$ is the fundamental group $\pi_1(Z, z)$.

Action on the fundamental groupoid

Suppose some group Γ continuously acts on the space Z. Then it acts on the fundamental groupoid $\Pi(Z)$ and hence on the category $Sh(\Pi(Z))$.

If Z is connected and $G = \pi_1(Z, z)$, then we get an action of the group Γ on the category $R(G)$, since $R(G) = M(G) = Sh(\Pi(Z))$.

This action is difficult to describe in the language of group theory.

Switching to other sites

Now move from the site of sets to some other sites.

Fix a **site** $S = (C, T)$, where C is a category and T a Grothendieck topology on it. Objects of S we will call "spaces" and denote by $X, Y, \ldots$.

Mostly we are interested in standard sites like Sets, Topological spaces (or some interesting subcategories of topological spaces), smooth manifolds, algebraic varieties over a field with Zariski topology or with some other topology.

Stacks – analogue of groupoids

Analogue of a groupoid in this case is a **stack** X over the site S.

How to define a stack X over the site S?

First remark that any object $Z \in C$ can be completely described in terms of the contravariant functor $F_Z : C^0 \to Sets$ given by $X \to Mor(X, Z)$ (Yoneda lemma).

So we can think about an object Z as a functor F of this type, satisfying some compatibility conditions.

In complete analogy, we define a **prestack** X over S as some kind of contravariant "pseudo-functor" $F : C^0 \to$ **Groupoids** , satlsfying some natural conditions.

Then, imposing on the prestack X some glueing conditions. we get a notion of a stack X over S.

Here are features of stacks that we are going to use:

(i) Every object $Z \in C$ can be considered as a stack.

(ii) Let X, Y be two stacks. Then $Mor(X, Y)$ is a groupoid.

In particular, every stack Y defines a pseudo-functor:
$F_Y : C^0 \rightarrow$ Groupoids. It is also defined by this pseudo-functor.

(iii) Let G be a group object in the site S. Then one can define the basic stack BG.

(iv) More generally, if Z is a G-space one can define the quotient stack $G \backslash Z$.

A system of sheaves on a site S is called a **fibered category** of sheaves; we denote such system as *Sheaves*(S).

For example, suppose S is the site of complex algebraic varieties. Given a variety X we can consider quasicoherent sheaves on X, or we can consider l-adic etale sheaves on X.

Also, we can pass to the underlying topological space $\tilde{X}$ with usual topology and consider sheaves on this space. For example, we can consider derived (or better ∞) category of constructible sheaves on $\tilde{X}$.

If we restrict attention to smooth varieties, we can work with sheaves of modules over the sheaf of smooth functions on the manifold $\tilde{X}$, and so on.

Suppose we fixed some fibered category *Sheaves*(S) over the site S and would like to extend it to stacks over S.

Consider a stack $\mathcal{Z}$. If we have some sheaf $\mathcal{F}$ over $\mathcal{Z}$, then for every space $X \in S$ and any morphism $v : X \to \mathcal{Z}$ we will get a sheaf $\mathcal{F}_{X,v} = v^*(\mathcal{F})$ on X. This collection of sheaves satisfies a variety of compatibility properties.

Now we define a sheaf $\mathcal{F}$ on the stack $\mathcal{Z}$ **to be** a collection of sheaves $\mathcal{F}_{X,v}$ satisfying these compatibility properties.

General definition of stacky G-modules

Suppose we fixed a site S and a system *Sheaves*(S) of sheaves on this site. Let G be a group object in S.

Then we define the category $\mathrm{M}(G)$ of **stacky G-modules** by $\mathrm{M}(G) := Sh(\mathrm{B}G)$

More generally, given a G-space Z we define the category $\mathrm{M}_G(Z)$ by $\mathrm{M}_G(Z) := Sh(\mathrm{G}\backslash Z)$.

One can think about this category as the category of G-equivariant sheaves on the space Z.

Representations of algebraic p-adic groups

Fix a p-adic field F and an algebraic group G over F.

The standard procedure in Representation theory is to consider the group $H = G(F)$ of F-points of G as a topological group.

Then one studies the category $R(G) := Rep_{sm}(H)$ of smooth representations of the topological group H.

Let us note that here we work with two sites – the site S of algebraic varieties over F and the site L of l-spaces.

If Z is an l-space, then sheaves on Z are just sheaves of complex vector spaces.

Claim. Let H be an l-group. i.e. a group object in the site L. Then the categories $\mathrm{R(H)} = \mathrm{M}(H) = Sh(\mathrm{B}H)$ are canonically equivalent to the category $Sh_H(pt)$ of H-equivariant sheaves on point.

Explicitly, R(H) is the category of **smooth** H-modules.

If Z is an H-space in L, then the category $\mathrm{M}_H(Z) = \mathrm{Sh(H\backslash Z)}$ is canonically equivalent to the category of H-equivariant sheaves on Z.

Stacky G-modules for algebraic groups

We have a functor of F–points, $F : S \to L$.
This functor extends to stacks $F : Stacks(S) \to Stacks(L)$.

Using this functor we define the system $Sheaves(S)$ of sheaves over S by $Sh(Z) := Sh(F(Z))$.

Then we extend sheaves to stacks over S.
It is easy to see that for any stack X over S we have Sh(X) = Sh(F(X))

Now we define the category $\mathrm{M}(G)$ of stacky G–modules to be the category $\mathrm{M}(G) := Sh(\mathrm{B}G) = Sh(F(\mathrm{B}G))$.

Stacky G-modules and G(F)-modules

We have $R(G) = R(H) = Sh(BH)$ $M(G) = Sh(BG) = Sh(F(BG))$

It turns out that in this case the category M(G) might be different from the category $R(G) = R(H) = Rep_{sm}(H)$.
The reason is that l-stacks BH and F(BG) **might be different.**

In fact, we will see that the l-stack $F(BG)$ is a union of several l-stacks of shape BH_i, where H_i are "pure inner forms" of the group H (usually finite number of them).

This implies that the category $M(G)$ of stacky G-modules is the product of categories $R(H_i)$.

25 Good illustration - Orthogonal group G

Let F be a p-adic field of characteristic 0, V be a vector space over F of dimension n.
Denote by Z the algebraic variety of non-degenerate quadratic forms on V.

Let us fix a point $q \in Z(F)$ and denote by G the orthogonal group $G = O(q)$. This is an algebraic group over F .

Let us describe the categories M(G) andR(G).
Consider the group $D = GL(V)$. This group transitively acts on the variety Z.

Going to F-points we see that Z(F) is the union of a finite number of open D-orbits Z_i .

We denote by Z_q the orbit of the point q.

Claim. (i) The category $\mathrm{M}(G)$ can be realizedas the category of $D(F)$-equivariant sheaves on the l-space $Z(F)$,
$\mathrm{M}(G) = Sh_{D(F)}(Z(F))$

(ii) The category $\mathrm{R}(G)$ can be realized as the category of $D(F)$-equivariant sheaves on the open orbit $Z_q \subset Z(F)$,
$\mathrm{R}(G) = Sh_{D(F)}(Z_q)$.

Note that the space $Z(F)$ is easy to describe explicitly (by some system of equations). On the other hand, the description of one particular orbit Z_q usually is quite involved.

From this I conclude, that the description of the category M(G) of equivariant sheaves on Z is probably much easier than that of the category R(G) of equivariant sheaves on the orbit Z_q .

I suspect that using the Langlands' correspondence we can classify the simple objects of M(G), i.e. simple equivariant sheaves on Z. However, it might be difficult to tell which of them belong to the subcategory R(G) (are supported on Z_q).

This means that the "natural" problem of classifying irreducible representations of the orthogonal l-group H = O(q,F) probably is not reasonable (and hence not interesting)

What technical tools we can use?

Tool I – Inside site L

Lemma. Let E be an l-group and $H \subset E$ be a closed subgroup. Consider the homogeneous E-space $X = E/H$.

Then the stacks $E\backslash X$ and $BH = H\backslash pt$ are canonically equivalent.

In particular, the category $R(H) = Sh_H(pt)$ can be equivalently described as the category $Sh_E(X)$.

Consider the functor of points $F : S \to L$ discussed above. Let us see how to extend it to quotient stacks.

Consider a quotient stack $\mathrm{X} = G\backslash Z$, where G is a linear algebraic group. We can try to define the l-stack $F(\mathrm{X})$ to be the l-stack $F(G)\backslash F(Z)$.

This does not work since the answer depends on a presentation of the stack X.

However, this does work for a class of groups G that are called **acyclic**.

We say that a group $G \in S$ is acyclic if any G torsor T is trivial.

As before, a G torsor T is a non-empty G space T such that the morphism $\gamma : G \times T \to T \times T$ is an isomorphism.

Given a stack $\mathfrak{X} = G\backslash Z \; over \; S$ we can embed the group G into some acyclic group D and consider a new presentation of the stack $\mathfrak{X}$

$$\mathfrak{X} = D\backslash Y, \text{ where } Y = D \times_G Z.$$

After this we define the l-stack $F(\mathfrak{X}) := F(D)\backslash F(Y)$.

It is easy to show that, up to canonical equivalence, this construction does not depend on choices.

Abundance of acyclic groups

By Hilbert 90 theorem, the group $D = GL(V)$ discussed above is acyclic.

Thus, we always can embed our linear algebraic group G into an acyclic group.

This is what we actually did above for an orthogonal group G.