Primes, Knots and the adele class space

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Toposes in Mondovi

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- Etale site of Spec $\mathbb{Z}$

- Analogy Knots and Primes

- Class Field theory

- Adele class space

- The arithmetic and scaling sites

- Main Theorem
- Drawing $\operatorname{Spec}\mathbb{Z}$ in $X_{\mathbb{Q}}$
- Finite covers of $X_{\mathbb{Q}}$
- Periodic orbit as elliptic curve
- K-theory of C^*-algebra

Grothendieck-Galois $\pi_1^{et}(X)$

Let X be a connected scheme. Then there exists a profinite group π, uniquely determined up to isomorphism, such that the category $\mathbf{FEt}_X$ of finite etale coverings of X is equivalent to the category π-sets of finite sets on which π acts continuously.

Separable algebras

B an A-algebra, and suppose that B is finitely generated and free as an A-module, $\mathrm{Tr}_{B/A} : B \to A$, A-linear map

$$\mathrm{Tr}_{B/A}(b) := \mathrm{Tr}\left(m_b\right), \quad m_b(x) = bx$$

$\phi : B \to \mathrm{Hom}_A(B, A)$ by

$$(\phi(x))(y) = \mathrm{Tr}(xy), \forall x, y \in B$$

B **separable** over A $\iff$ ϕ is an isomorphism.

Exo : $B = A[x]/\left(x^2\right)$ is not separable.

Finite etale morphism

A morphism $f : Y \to X$ of schemes is finite étale if there exists a covering of X by open affine subsets $U_i = \operatorname{Spec} A_i$, such that for each i the open subscheme $f^{-1}(U_i)$ of Y is affine, and equal to $\operatorname{Spec} B_i$, where B_i is a free separable A_i-algebra.

Ramification

Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field K, and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. For a field extension L/K we can consider the ring of integers $\mathcal{O}_L$ (which is the integral closure of $\mathcal{O}_K$ in L), and the ideal $\mathfrak{p}\mathcal{O}_L$ of $\mathcal{O}_L$. This ideal may or may not be prime, but for finite $[L : K]$, it has a factorization into prime ideals :

$$\mathfrak{p} \cdot \mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}$$

where the $\mathfrak{p}_i$ are distinct prime ideals of $\mathcal{O}_L$. Then $\mathfrak{p}$ is said to ramify in L if $e_i > 1$ for some i ; otherwise it is unramified.

π_1^{et} and Galois

Let X be a normal integral scheme, K its function field, $\bar{K}$ an algebraic closure of K, and M the composite of all finite separable field extensions L of K with $L \subset \bar{K}$ for which X is unramified in L. Then the fundamental group $\pi_1^{et}(X)$ is isomorphic to the Galois group Gal(M/K).

Sphere S^3	Scheme Spec $\mathbb{Z}$
$\pi_1\left(S^3\right) = \{1\}$	$\pi_1^{et}(\text{Spec}\,\mathbb{Z}) = \{1\}$ Kronecker-Minkowski
$H^3\left(S^3, \mathbb{Z}\right) = \mathbb{Z}$	$H^3\,(\text{Spec}\,\mathbb{Z}, \mathbf{G}_m) = \mathbb{Q}/\mathbb{Z}$ Artin-Verdier
Knot C	Prime p Mumford Mazur 1963

Knot C	Prime p
Inclusion $C \subset S^3$	$r^* : \operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$
Knot complement $X = S^3 - C$	$\operatorname{Spec} \mathbb{Z} \backslash \{p\}$
$\pi_1 \left(S^3 - C\right)^{ab} = \mathbb{Z}$	$\pi_1^{et}(\operatorname{Spec} \mathbb{Z}[\frac{1}{p}])^{ab} = \mathbb{Z}_p^*$
$\pi_1 (C_1) \to \pi_1 \left(S^3 - C_2\right)^{ab}$	$\pi_1^{et} (\operatorname{Spec} \mathbb{F}_p) \to \pi_1^{et}(\operatorname{Spec} \mathbb{Z}[\frac{1}{q}])^{ab}$
Linking Number (C_1, C_2)	$p \in \mathbb{Z}_q^*$

Class Field Theory

L'objet de la théorie du corps de classes est de montrer comment les extensions abéliennes d'un corps de nombres algébriques K peuvent être déterminées par des éléments tirés de la connaissance de K lui-même ; ou, si l'on veut présenter les choses en termes dialectiques, comment un corps possède en soi les éléments de son propre dépassement.

C. Chevalley (1940)

$$\mathsf{Gal}(K^{ab} : K) \simeq \left(\mathsf{GL}_1(\mathbb{A}_K)/K^\times\right) / \left(\mathsf{GL}_1(\mathbb{A}_K)/K^\times\right)_0$$

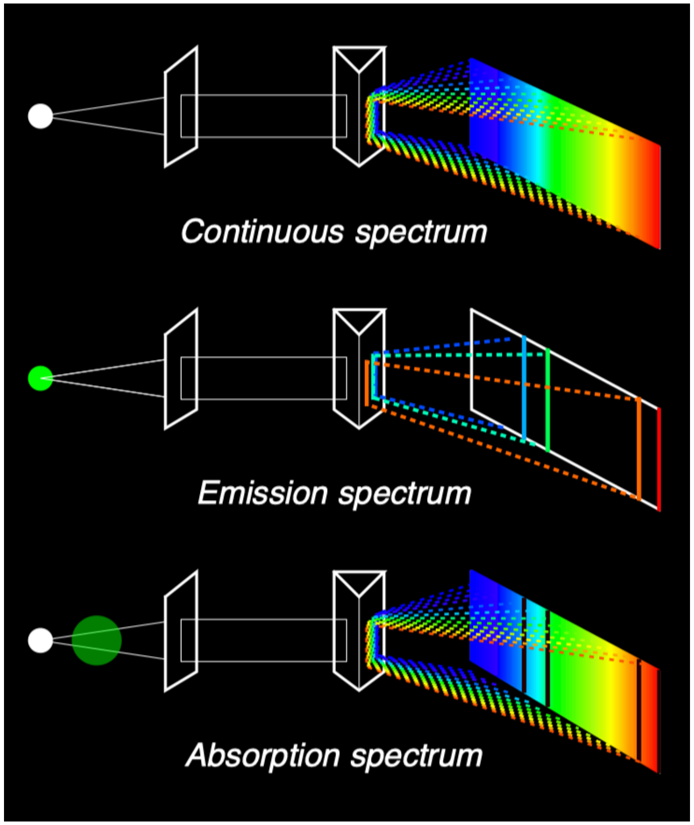
Adele class space $\mathbb{A}_K/K^\times$, ac 1996

▶ Local to Global : $K_v^\times \subset \mathrm{GL}_1(\mathbb{A}_K)/K^\times$
Isotropy subgroup of adele classes with a zero at the place v

▶ Explicit formulas, $K_v^\times$ acting on transverse space K_v

$$\int k(x,x)dx = \int \delta(x - \lambda x)dx = \frac{1}{|1-\lambda|}$$

▶ Spectral realization = absorption spectrum



Adele class space $\to$ scaling site

(ac+cc, 2014)

NCG $\Longleftrightarrow$ Topos

$$X_{\mathbb{Q}}^{ab} = \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}$$

$$\Downarrow \pi$$

$$X_{\mathbb{Q}} = \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^{*}$$

The space $X_{\mathbb{Q}}$

- Adelic $X_{\mathbb{Q}} = \mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^{*}$
- Rank one subgroups of $\mathbb{R}$ (and also up to $\sim$)
- Points of arithmetic site over $\mathbb{R}_{+}^{\max}$
- Points of topos $[0, \infty) \rtimes \mathbb{N}^{\times}$ (scaling site)

Rank one subgroups of $\mathbb{R}$

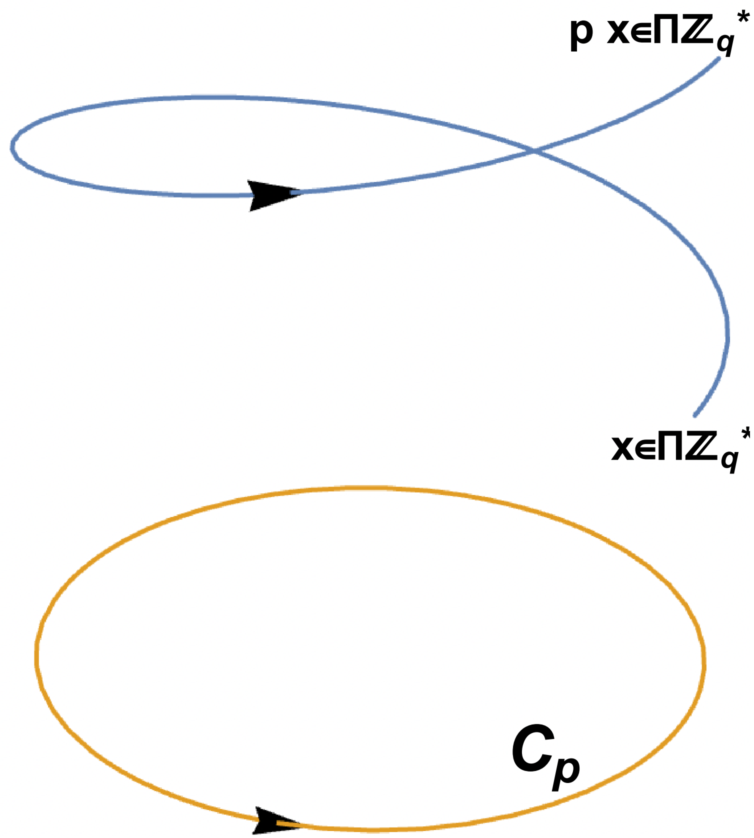
Let $\mathbb{A}^f$ be the ring of finite adeles of $\mathbb{Q}$. Let Φ be the map from $\left(\mathbb{A}^f/\hat{\mathbb{Z}}^*\right) \times \mathbb{R}_+^*$ to subgroups of $\mathbb{R}$ defined by

$$\Phi(a, \lambda) := \lambda H_a, \quad H_a := \{q \in \mathbb{Q} \mid aq \in \hat{\mathbb{Z}}\}.$$

Then Φ is a bijection between the subset of $X_{\mathbb{Q}} = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^*$ formed of adele classes with non-zero archimedean component, and the set of non-zero subgroups of $\mathbb{R}$ whose elements are pairwise commensurable.

Main Theorem (ac+cc)

Let p be a prime. Let $\{\mathrm{Frob}_p\} \in \pi_1^{et}(\mathrm{Spec}\,(\mathbb{F}_p))$ be the canonical generator. The inverse image $\pi^{-1}(C_p) \subset X_{\mathbb{Q}}^{ab}$ of the periodic orbit C_p is canonically isomorphic to the mapping torus of the multiplication by $r^*\{\mathrm{Frob}_p\}$ in the abelianized étale fundamental group $\pi_1^{et}(\mathrm{Spec}\,\mathbb{Z}_{(p)})^{ab}$. The canonical isomorphism is equivariant for the action of the idele class group.


p x∈ΠZ_q*
x∈ΠZ_q*
C_p

Periodic orbit C_p

$S = \{p, \infty\}$

$$\pm\, p^{\mathbb{Z}} \backslash \left(\mathbb{Q}_p \times \mathbb{R}\right) / \mathbb{Z}_p^* = X_{\mathbb{Q},S}$$

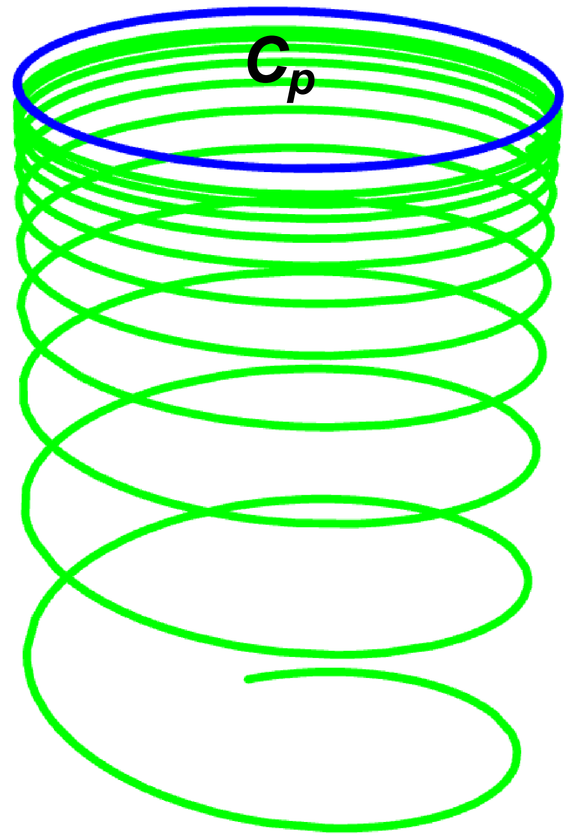
Elements of $\left(\mathbb{Q}_p \times \mathbb{R}\right) / \mathbb{Z}_p^*$ are pairs (p^n, λ) with $p^\infty = 0$.
The group $\mathbb{R}_+^*$ acts on $X_{\mathbb{Q},S}$,

Generic orbit : Free action on pairs with elts $\neq 0$.

Periodic orbit : $(0, \lambda) \sim (0, p\lambda) \to C_p = p^{\mathbb{Z}} \backslash \mathbb{R}_+^*$

$$(p^n, \lambda) \to (0, \lambda) \;\&\; (p^n, \lambda) \sim (1, p^{-n}\lambda)$$

Generic orbit dense in C_p



Several primes

Finite set of places $S \ni \infty$, $p_j \in S$,

$$X_{\mathbb{Q},S} := \Gamma \backslash \left(\prod_S \mathbb{Q}_v \right) / \prod_{S \backslash \infty} \mathbb{Z}_p^*$$

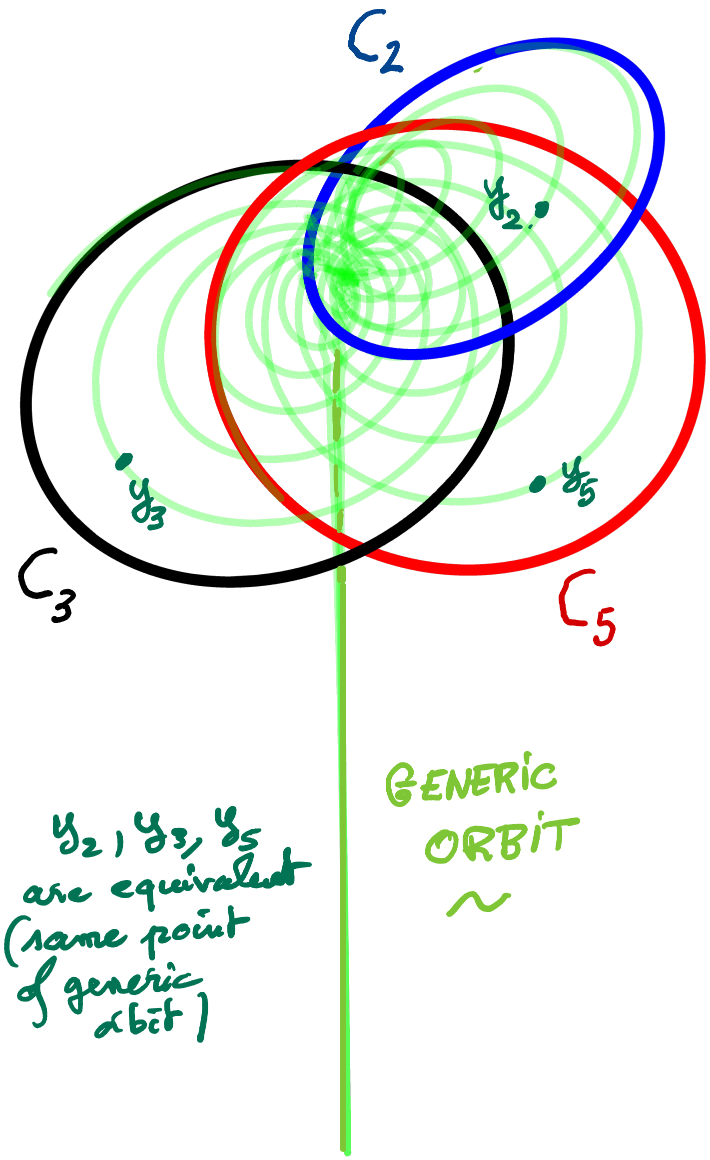
where

$$\Gamma := \{\pm p_1^{n_1} \dots p_k^{n_k} \mid n_j \in \mathbb{Z}\}$$

The group $\mathbb{R}_+^*$ acts on $X_{\mathbb{Q},S}$,

Generic orbit : Free action on x with $x_v \neq 0$ for all v.

Periodic orbits : $x_p = 0$ & $x_v \neq 0 \to C_p = p^{\mathbb{Z}} \backslash \mathbb{R}_+^*$


C2
y2
C3
y3
y5
C5
GENERIC ORBIT
~
y2, y3, y5 are equivalent (same point of generic orbit)

Etale Facts

— The abelianized étale fundamental group $\pi_1^{et}(\text{Spec } \mathbb{Z}_{(p)})^{ab}$ is canonically isomorphic to $\prod_{q \neq p} \mathbb{Z}_q^*$.

— The image $\pi_1^{et}(r^*)\,\{\text{Frob}_p\}$ in the étale abelianized fundamental group $\pi_1^{et}(\text{Spec } \mathbb{Z}_{(p)})^{ab} \simeq \prod_{q \neq p} \mathbb{Z}_q^*$ is equal to p diagonally embedded in $\prod_{q \neq p} \mathbb{Z}_q^*$.

The maximal abelian extension of $\mathbb{Q}$ in which p is unramified is obtained by adjoining all roots of unity of order prime to p following the local to global proof of the Kronecker-Weber theorem. Its Galois group is $\prod_{q \neq p} \mathbb{Z}_q^*$. The action of Frob_p on roots of unity is given by raising to the power p.

Abelian finite covers of $X_{\mathbb{Q}}$

L finite abelian extension of $\mathbb{Q} \to$ finite cover $X_{\mathbb{Q}}^L \to X_{\mathbb{Q}}$:

$$X_{\mathbb{Q}}^L := \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / W, \quad W \subset \hat{\mathbb{Z}}^*, \; W = \mathrm{Ker}\left(\hat{\mathbb{Z}}^* \to \mathrm{Gal}(L/\mathbb{Q})\right)$$

$$\pi^L : X_{\mathbb{Q}}^L \to X_{\mathbb{Q}}$$

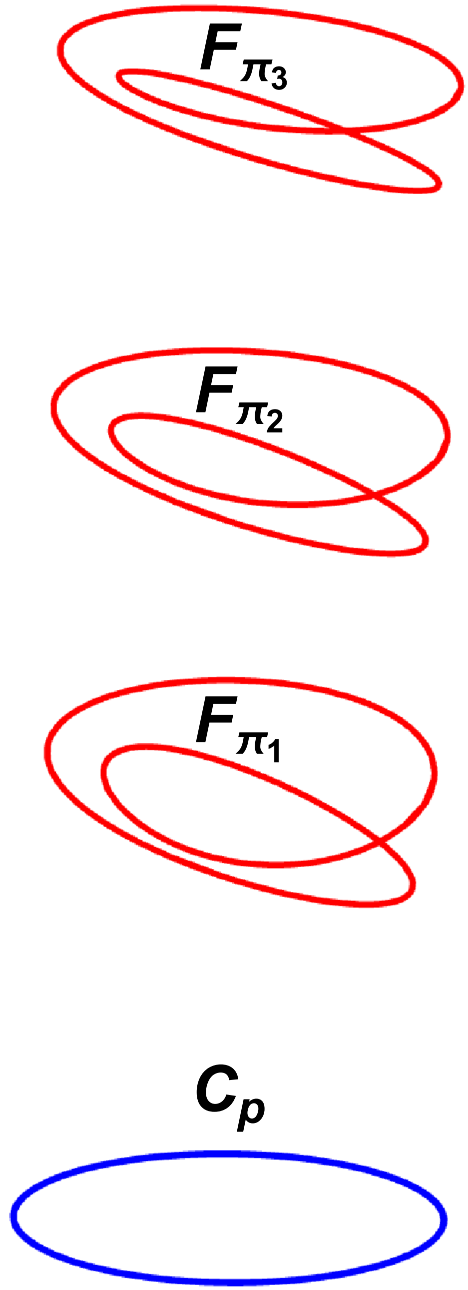
The group $G = \mathrm{Gal}(L/\mathbb{Q})$ acts transitively on each fiber, and we say that the cover is unramified at $x \in X_{\mathbb{Q}}$ when G acts freely in the fiber $\{y, \pi^L(y) = x\}$.
There exists a smallest set R of places such that

$$x \in X_{\mathbb{Q}}, \; x_v \neq 0 \; \forall v \in R \Rightarrow X_{\mathbb{Q}}^L \text{ unramified at } x$$

Theorem

$L \to \left(\pi^L : X^L_{\mathbb{Q}} \to X_{\mathbb{Q}}\right)$ is a contravariant functor and

1. The finite set R of places at which the cover ramifies is the union of the archimedean place with the set of primes at which L ramifies.
2. Let $p \notin R$ then the monodromy of C_p in $X^L_{\mathbb{Q}}$ is the element of G given by the Frobenius Frob_p.
3. The connected components of the inverse image of C_p are circles labeled by the places of L over the prime p.


F_{π_3}
F_{π_2}
F_{π_1}
C_p

Role of η for $\mathrm{Spec}\,\mathbb{Z}$

Let $j : \eta \to \mathrm{Spec}\,\mathbb{Z}$ be the generic point, the following sheaves on $\mathrm{Spec}\,\mathbb{Z}$ form an exact sequence

$$0 \to \mathbf{G}_m \to j_*\mathbf{G}_{m,\eta} \to \coprod_p \mathbb{Z}|p \to 0,$$

$$\mathrm{H}^q\left(\mathrm{Spec}\,\mathbb{Z}, \coprod_p \mathbb{Z}|p\right) = 0 \qquad \text{for } q = 1, q > 2.$$

$$\mathrm{H}^2\left(\mathrm{Spec}\,\mathbb{Z}, \coprod_p \mathbb{Z}|p\right) = \bigoplus_p \mathbb{Q}/\mathbb{Z}$$

$$0 \to \mathrm{H}^2(\mathrm{Spec}\,\mathbb{Z}, \mathbf{G}_m) \to \mathrm{H}^2(\eta, \mathbf{G}_{m,\eta}) \xrightarrow{r_1} \bigoplus_p \mathbb{Q}/\mathbb{Z}$$
$$\to \mathrm{H}^3(\mathrm{Spec}\,\mathbb{Z}, \mathbf{G}_m) \to \mathrm{H}^3(\eta, \mathbf{G}_{m,\eta}) \to 0.$$

Spectral realization as $H^1(X_{\mathbb{Q}}^{ab}, \eta)$

The idele class group acts on

$$H^1(X_{\mathbb{Q}}^{ab}, \eta)$$

The map $\mathcal{E} : \mathcal{S}(\mathbb{A}_{\mathbb{Q}})_0 \to \mathcal{S}(C_{\mathbb{Q}})$ comes from the trace map in Hochschild homology using cross products

$$\mathcal{S}(\mathbb{A}_{\mathbb{Q}}) \ltimes \mathbb{Q}^{\times}$$

$$0 \to H^0(X, Y) \to H^0(X) \xrightarrow{\mathcal{E}} H^0(Y) \to H^1(X, Y)$$

$$\to H^1(X) \xrightarrow{\rho} H^1(Y) \to H^2(X, Y) \to \ldots$$

Geometric structure of $X_{\mathbb{Q}}$

The action of $\mathbb{R}_+^\times$ on the space $X_{\mathbb{Q}}$ is in fact the action of the Frobenius automorphisms Fr_λ on the points of the arithmetic site over $\mathbb{R}_+^{\max}$.

Topos + characteristic 1

— Arithmetic Site.

— Frobenius correspondences.

— Extension of scalars to $\mathbb{R}_+^{\max}$.

Why semirings ?

A category $\mathcal{C}$ is *semiadditive* if it has finite products and corpoducts, the morphism $0 \to 1$ is an isomorphism (thus $\mathcal{C}$ has a 0), and the morphisms

$$\gamma_{M,N} : M \vee N \to M \times N$$

are isomorphisms.
Then End(M) is naturally a semiring for any object M.

Finite semifields, characteristic 1

$\mathbb{K} =$ finite semifield : then $\mathbb{K}$ is a field or $\mathbb{K} = \mathbb{B}$:

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

The semifield $\mathbb{Z}_{\max}$

Lemma : Let F be a semifield of characteristic 1, then for $n \in \mathbb{N}^{\times}$ the map $\mathrm{Fr}_n \in \mathrm{End}(\mathbb{F})$, $\mathrm{Fr}_n(x) := x^n \ \forall x \in F$ defines an **injective endomorphism** of F.

$\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$, unique semifield with multiplicative group infinite cyclic.
multiplicative notation : Addition $\vee$, $u^n \vee u^m = u^k$, *with* $k = \sup(n, m)$. *Multiplication :* $u^n u^m = u^{n+m}$.

Map $\mathbb{N}^{\times} \to \mathrm{End}(\mathbb{Z}_{\max})$, $n \mapsto \mathrm{Fr}_n$ is isomorphism of semigroups. (extend to 0)

Arithmetic Site $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$

$\mathbb{Z}_{\max}$ on which $\mathbb{N}^\times$ acts by $n \mapsto \mathrm{Fr}_n$ is a semiring in the topos $\widehat{\mathbb{N}^\times}$ of sets with an action of $\mathbb{N}^\times$.

The *Arithmetic Site* $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ is the topos $\widehat{\mathbb{N}^\times}$ endowed with the *structure sheaf* : $\mathcal{O} := \mathbb{Z}_{\max}$ semiring in the topos.

Characteristic 1

The role of $\mathbb{F}_q$ is played by

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

No finite extension, but

$\mathrm{Fr}_\lambda(x) = x^\lambda$ automorphisms of $\mathbb{R}_+^{\max}$.

$$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max}) = \mathbb{R}_+^\times$$

Points of the arithmetic site over $\mathbb{R}_+^{\max}$

These are defined as pairs $(p, f_p^\#)$ of a point p of $\widehat{\mathbb{N}^\times}$ and local morphism $f_p^\# : \mathcal{O}_p \to \mathbb{R}_+^{\max}$.

Theorem

The points $\mathscr{A}(\mathbb{R}_+^{\max})$ of $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ on $\mathbb{R}_+^{\max}$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$. The action of the Frobenius Fr_λ of $\mathbb{R}_+^{\max}$ corresponds to the action of the idele class group.

Extension of scalars to $\mathbb{R}_{\max}$

The following holds :

$$\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max} \simeq \mathcal{R}(\mathbb{Z})$$

$\mathcal{R}(\mathbb{Z})$ = semiring of continuous, convex, piecewise affine functions on $\mathbb{R}_+$ with slopes in $\mathbb{Z} \subset \mathbb{R}$ and only finitely many discontinuities of the derivative

These functions are endowed with the pointwise operations of functions with values in $\mathbb{R}_{\max}$

Points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$

Theorem : The points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$.

Corollary : One has a canonical isomorphism Θ between the points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$ and $\mathscr{A}(\mathbb{R}_+^{\max})$ *i.e.* the points of the arithmetic site defined over $\mathbb{R}_+^{\max}$.

Structure sheaf of $[0,\infty) \rtimes \mathbb{N}^{\times}$

This is the sheaf on $[0,\infty) \rtimes \mathbb{N}^{\times}$ associated to convex, piecewise affine functions with integral slopes

Same as for the localization of zeros of analytic functions $f(X) = \sum a_n X^n$ in an annulus

$$A(r_1, r_2) = \{z \in K \mid r_1 < |z| < r_2\}$$

$$\tau(f)(x) := \max_n \{-nx - v(a_n)\}\,, \quad \forall x \in (-\log r_2, -\log r_1)$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta$$

Periodic orbit as elliptic curve

Analogue of $\mathbb{C}/q^{\mathbb{Z}}$

$$C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$$

The degree of a divisor is a real number. For any divisor D in C_p, there is a corresponding Riemann-Roch problem with solution space $H^0(D)$. The continuous dimension * $\mathrm{Dim}(H^0(D))$ of this $\mathbb{R}_+^{\max}$-module is defined by the limit

$$\mathrm{Dim}(H^0(D)) := \lim_{n\to\infty} p^{-n}\dim(H^0(D)^{p^n}) \qquad (1)$$

$*$. In analogy with von-Neumann's continuous dimensions of the theory of type II factors where $H^0(D)^{p^n}$ is a naturally defined filtration and $\dim(\mathcal{E})$ denotes the topological dimension of an $\mathbb{R}_+^{\max}$-module.

(i) Let $D \in \mathrm{Div}(C_p)$ be a divisor with $\deg(D) \geq 0$. Then the limit in (1) converges and one has

$$\mathrm{Dim}(H^0(D)) = \deg(D).$$

(ii) The following Riemann-Roch formula holds

$$\mathrm{Dim}(H^0(D)) - \mathrm{Dim}(H^0(-D)) = \deg(D) \qquad \forall D \in \mathrm{Div}(C_p).$$

Rational functions

For $W \subset C_p$ open, $\mathcal{O}_p(W)$ is simplifiable, one lets $\mathcal{K}_p$ the sheaf associated to the presheaf $W \mapsto \text{Frac}\,\mathcal{O}_p(W)$.

Lemma The sections of the sheaf $\mathcal{K}_p$ are continuous piecewise affine functions with slopes in H_p endowed with max ($\vee$) and the sum.

$$(x-y)\vee(z-t) = ((x+t)\vee(y+z)) - (y+t).$$

Cartier divisors

Lemma : The sheaf CDiv(C_p) of Cartier divisors *i.e.* the quotient sheaf $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, is isomorphic to the sheaf of naive divisors $H \mapsto D(H) \in H$,

$$\forall \lambda,\ \exists V \text{ open } \lambda \in V,\ D(\mu) = 0\,,\ \forall \mu \in V,\ \mu \neq \lambda$$

Point $\mathfrak{p}_H$ associated to $H \subset \mathbb{R}$ and f section of $\mathcal{K}$ at $\mathfrak{p}_H$.

$$\text{Order}(f) = h_+ - h_- \in H \subset \mathbb{R}$$

$$h_\pm = \lim_{\epsilon \to 0\pm} \frac{f((1+\epsilon)H) - f(H)}{\epsilon}$$

.

Divisors

Definition : A divisor is a global section of $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, *i.e.* a map $H \to D(H) \in H$ vanishing except on finitely many points.

Proposition : (i) The divisors Div(C_p) form an abelian group under addition.
(ii) The condition $D'(H) \geq D(H)$, $\forall H \in C_p$, defines a partial order on Div(C_p).
(iii) The **degree** map is additive and order preserving

$$\text{deg}(D) := \sum D(H) \in \mathbb{R}.$$

Principal divisors

The sheaf $\mathcal{K}_p$ admits global sections :

$$\mathcal{K} := \mathcal{K}(C_p) = H^0(\mathbb{R}_+^*/p^{\mathbb{Z}}, \mathcal{K}_p)$$

the semifield of global sections.

Principal divisors : The map which to $f \in \mathcal{K}^\times$ associates the divisor

$$(f) := \sum_H (H, \mathrm{Ord}_H(f)) \in \mathrm{Div}(C_p)$$

is a group morphism $\mathcal{K}^\times \to \mathcal{P} \subset \mathrm{Div}(C_p)$.

The subgroup $\mathcal{P} \subset \mathrm{Div}(C_p)$ of principal divisors is **contained in the kernel** of the morphism $\deg : \mathrm{Div}(C_p) \to \mathbb{R}$:

$$\sum_H \mathrm{Ord}_H(f) = 0\,, \;\; \forall f \in \mathcal{K}^\times.$$

Invariant χ

For $p > 2$ one considers the ideal $(p-1)H_p \subset H_p$.

$$0 \to (p-1)H_p \to H_p \xrightarrow{r} \mathbb{Z}/(p-1)\mathbb{Z} \to 0$$

Lemma : For $H \subset \mathbb{R}$, $H \simeq H_p$, the map $\chi : H \to \mathbb{Z}/(p-1)\mathbb{Z}$, $\chi(\mu) = r(\mu/\lambda)$ where $H = \lambda H_p$ is independent of the choice of λ.

Theorem

The map $(\deg, \chi)$ is a **group isomorphism**

$$(\deg, \chi) : \mathrm{Div}(C_p)/\mathcal{P} \to \mathbb{R} \times (\mathbb{Z}/(p-1)\mathbb{Z})$$

where $\mathcal{P}$ is the subgroup of principal divisors.

Theta Functions on $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$

$$\prod_0^\infty (1 - t^m w) \to f_+(\lambda) := \sum_0^\infty (0 \vee (1 - p^m \lambda))$$

$$\prod_1^\infty (1 - t^m w^{-1}) \to f_-(\lambda) := \sum_1^\infty \Big(0 \vee (p^{-m}\lambda - 1)\Big)$$

Theorem

Any $f \in \mathcal{K}(C_p)$ has a canonical decomposition

$$f(\lambda) = \sum_i \Theta_{h_i,\mu_i}(\lambda) - \sum_j \Theta_{h'_j,\mu'_j}(\lambda) - h\lambda + c$$

where $c \in \mathbb{R}$, $(p-1)h = \sum h_i - \sum h'_j$ and $h_i \leq \mu_i < p h_i$, $h'_j \leq \mu_j < p h'_j$.

p-adic filtration $H^0(D)^\rho$

Definition : Let $D \in \mathrm{Div}(C_p)$ one lets

$$H^0(D) := \{f \in \mathcal{K}(C_p) \mid D + (f) \geq 0\}$$

It is an $\mathbb{R}_{\max}$-module, $f, g \in H^0(D) \Rightarrow f \vee g \in H^0(D)$.

Lemma : Let $D \in \mathrm{Div}(C_p)$ be a divisor, one gets a filtration of $H^0(D)$ by $\mathbb{R}_{\max}$-sub-modules :

$$H^0(D)^\rho := \{f \in H^0(D) \mid \|f\|_p \leq \rho\}$$

using the p-adic norm.

Real valued Dimension

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n\to\infty} p^{-n}\mathrm{dim}_{\mathrm{top}}(H^0(D)^{p^n})$$

where the *topological dimension* $\mathrm{dim}_{\mathrm{top}}(X)$ is the number of real parameters on which solutions depend.

Riemann-Roch Theorem

(i) Let $D \in \mathrm{Div}(C_p)$ a divisor with $\deg(D) \geq 0$, then

$$\lim_{n\to\infty} p^{-n}\mathrm{dim}_{\mathrm{top}}(H^0(D)^{p^n}) = \deg(D)$$

(ii) One has the Riemann-Roch formula :

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) - \mathrm{Dim}_{\mathbb{R}}(H^0(-D)) = \deg(D)\,,\ \forall D \in \mathrm{Div}(C_p).$$

Questions

▶ Are abelian covers of Spec $\mathbb{Z}$ sufficient for the 3-dimensionality ?

▶ Explore the etale site over the scaling site.

- Cover of C_p viewed as a tropical elliptic curve, with $\mathbb{Q}_p \times \mathbb{R}$ involved in Riemann-Roch formula.

- Can one characterize the finite covers as finite abelian tropical covers of the scaling site endowed with its structure sheaf.

K-theory of C^*-algebra

The simplest meaningful computation of the K-theory of the involved C^*-algebras is for the cross product A associated to the union in $X_{\mathbb{Q},S}$, $S = \{p, q, \infty\}$, of the generic orbit with the three periodic orbits C_p, C_q, C_∞. One obtains that $K_0(A) \simeq \mathbb{Z}^3$ reflects the presence of the three periodic orbits, while $K_1(A) \simeq \mathbb{Z}^2$ reflects the one-dimensionality of the periodic orbits C_p, C_q.

$$\text{NCG} \iff \text{Topos}$$

K-theory of C^*-algebra

$S = \{p, q, \infty\}$, and the open subspace

$$\Omega \subset \mathbb{A}_{\mathbb{Q},S} = \mathbb{Q}_p \times \mathbb{Q}_q \times \mathbb{R}$$

given by the adeles which have at most one zero.

Dividing Ω by the action of the compact group $G = \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ gives a locally compact space $Z := \Omega/G$ which is the union of the following 4 subspaces :

1. $Z_\emptyset = \left(\mathbb{Q}_p^* \times \mathbb{Q}_q^* \times \mathbb{R}^*\right)/G \simeq p^{\mathbb{Z}} \times q^{\mathbb{Z}} \times \{\pm 1\} \times \mathbb{R}_+^*$.
2. $Z_p = \left(\{0\} \times \mathbb{Q}_q^* \times \mathbb{R}^*\right)/G \simeq \{0\} \times q^{\mathbb{Z}} \times \{\pm 1\} \times \mathbb{R}_+^*$.
3. $Z_q = \left(\mathbb{Q}_p^* \times \{0\} \times \mathbb{R}^*\right)/G \simeq p^{\mathbb{Z}} \times \{0\} \times \{\pm 1\} \times \mathbb{R}_+^*$.
4. $Z_\infty = \left(\mathbb{Q}_p^* \times \mathbb{Q}_q^* \times \{0\}\right)/G \simeq p^{\mathbb{Z}} \times q^{\mathbb{Z}} \times \{0\}$.

$A = C_0(Z) \ltimes \Gamma$, $\Gamma = \{\pm p^n q^m \mid n, m \in \mathbb{Z}\}$

The cross product C^*-algebras are, up to Morita equivalence, with $\mathcal{K}$ the compact operators,

1. $A_\emptyset = C_0(Z_\emptyset) \ltimes \Gamma = \mathcal{K} \otimes C_0(\mathbb{R}^*_+)$.
2. $A_p = C_0(Z_p) \ltimes \Gamma = \mathcal{K} \otimes C(C_p)$
3. $A_q = C_0(Z_q) \ltimes \Gamma = \mathcal{K} \otimes C(C_q)$
4. $A_\infty = C_0(Z_\infty) \ltimes \Gamma = \mathcal{K} \otimes C^*(\mathbb{Z}/2\mathbb{Z})$.

None of the involved C^*-algebras is unital but the C^*-algebra $B = A_p \oplus A_q \oplus A_\infty$ on the right is Morita equivalent to the unital C^*-algebra $\mathbf{C} := C(C_p) \oplus C(C_q) \oplus C^*(\mathbb{Z}/2\mathbb{Z})$. We thus get the exact hexagon of K-theory groups

$$\begin{array}{ccccc} & K_0(A_\emptyset) & \xrightarrow{\iota_*} & K_0(A) & \\ \delta_1 \nearrow & & & & \searrow \rho_* \\ K_1(B) & & & & K_0(B) \\ \rho_* \nwarrow & & & & \swarrow \delta_0 \\ & K_1(A) & \xleftarrow{\iota_*} & K_1(A_\emptyset) & \end{array} \qquad (2)$$

One has

— $K_0(A_\emptyset) = K_0(C_0(\mathbb{R}_+^*)) = \{0\}$.

— $K_1(A_\emptyset) = K_1(C_0(\mathbb{R}_+^*)) = \mathbb{Z}$.

— $K_0(A_p) = K_0(C(C_p)) = \mathbb{Z}$,

— $K_1(A_p) = K_1(C(C_p)) = \mathbb{Z}$.

— $K_0(A_q) = K_0(C(C_q)) = \mathbb{Z}$,

— $K_1(A_q) = K_1(C(C_q)) = \mathbb{Z}$.

— $K_0(A_\infty) = K_0(C^*(\mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}^2$,

— $K_1(A_\infty) = K_1(C^*(\mathbb{Z}/2\mathbb{Z})) = \{0\}$.

$$\begin{array}{ccccccc} & & K_0(A_\emptyset)=\{0\} & \xrightarrow{\iota_*} & K_0(A) & & \\ & \nearrow^{\delta_1} & & & & \searrow^{\rho_*} & \\ K_1(B)=\mathbb{Z}^2 & & & & & & K_0(B)=\mathbb{Z}^4 \\ & \nwarrow_{\rho_*} & & & & \swarrow_{\delta_0} & \\ & & K_1(A) & \xleftarrow[\iota_*]{} & K_1(A_\emptyset)=\mathbb{Z} & & \end{array} \qquad (3)$$

One shows that the map $\delta_0 : K_0(B) \to K_1(A_\emptyset)$ is surjective.