The Grothendieck topos of generalized smooth functions

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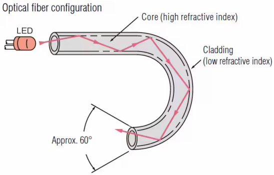
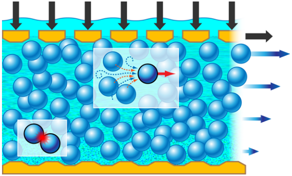
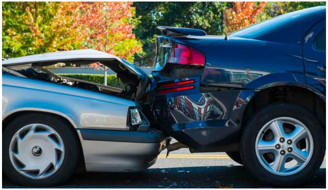
Toposes in Mondovì 2024

Abrupt changes, nonlinear systems and impulsive stimuli

Nature is full of non-smooth systems:







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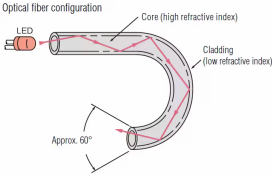

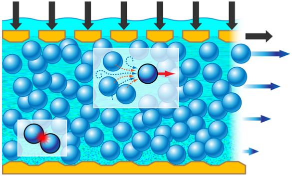

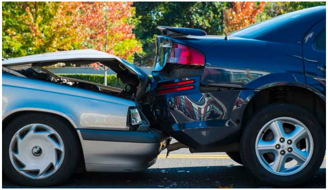

What kind of mathematical model? Piecewise smooth? Infinite derivatives in infinitesimal intervals? Differential equations or not? Calculus of variations?

Also within mathematics

- L. Hörmander: "*In differential calculus one encounters immediately the unpleasant fact that not every function is differentiable. The purpose of distribution theory is to remedy this flaw; indeed, the space of distributions is essentially the smallest extension of the space of continuous functions where differentiability is always well defined*"

- V.I. Arnol'd: "*Nowadays, when teaching analysis, it is not very popular to talk about infinitesimal quantities. Consequently, present-day students are not fully in command of this language. Nevertheless, it is still necessary to have command of it*"

The simplest way to differentiate continuous functions

Theorem

The space of Schwartz distributions $(\mathcal{D}', \lambda, (D_i)_i)$ *on* $\mathbb{R}^n$ *is a co-universal solution (the unique arrow starts from the solution) of the following problem:*

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3. $D_k : \mathcal{D}' \longrightarrow \mathcal{D}'$ *(derivative w.r.t.* x_k*),* $k = 1, \ldots, n$*, are compatible with partial derivatives of* $\mathcal{C}^1_k$ *(*k *means "w.r.t.* x_k*") functions:*

$$\begin{array}{ccccc}
\mathcal{C}^1_k & \xhookrightarrow{\iota_k} & \mathcal{C}^0 & \xrightarrow{\lambda} & \mathcal{D}' \\
 & \searrow^{\frac{\partial}{\partial x_k}} & & & \downarrow{\scriptstyle D_k} \\
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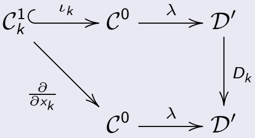

4. *Let* $\alpha \in \mathbb{N}^n$, $f \in \mathcal{C}^0(U)$ *and* $U = (c_1 - r, c_1 + r) \times \overset{n}{\ldots\ldots} \times (c_n - r, c_n + r)$. *Assume* $f = \theta_1 + \ldots + \theta_n$*, where* θ_k *is a polynomial in* x_k *of degree* $< \alpha_k$ *whose coefficients* $\in \mathcal{C}^0(U)$ *independent by* x_k*, then* $D^\alpha_U(\lambda_U(f)) = 0$ *(*$\mathcal{P}_\alpha$ *denotes the set of these polynomials)*

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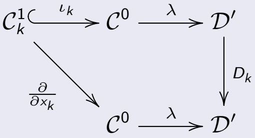

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5. $D_h \circ D_k = D_k \circ D_h$ *for all* $h, k = 1, \ldots, n$

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- Nonlinear ODE or PDE cannot even be formulated $y' = F(t, y)$
- Even for linear ODE we cannot formulate an initial value problem $\nexists\, y(t_0)$

Grothendieck's freedom from peers

1. The anchoring bias is the tendency to rely too heavily on one trait or piece of information when making decisions (usually the first piece of information acquired on that subject)
2. Common source bias, the tendency to combine or compare research studies from the same source, or from sources that use the same methodologies or data
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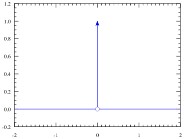
Grothendieck called this kind of behavior in math
"lacking of freedom from peers"

Different point of views on generalized functions

(Schwartz, Lojasiewicz, Laugwitz, Schmieden, Egorov, Robinson, Colombeau, Rosinger, Levi-Civita, Keisler, Connes, etc.)



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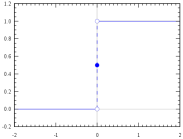
$$T(\varphi) = \int_{\mathbb{R}^n} T(x)\varphi(x)\,\mathrm{d}x$$



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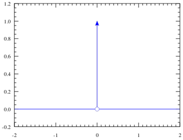

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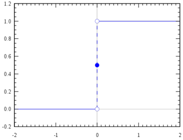

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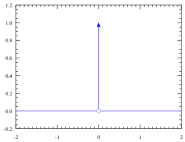

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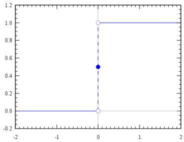

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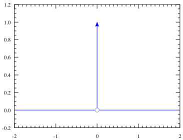

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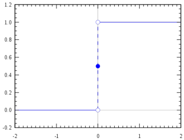

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- E.g. set an infinitesimal σ in a Gaussian density $N(x,\sigma)$ to have a δ-like function

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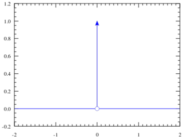

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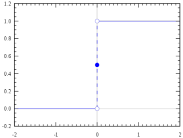

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To formalize this original approach, we need infinitesimal and infinite numbers among our new extended ring of scalars $\bar{\mathbb{R}} \supseteq \mathbb{R}$

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   e.g. $B_{\mathrm{d}\rho^q}(c)$ for $q \in \mathbb{N}$

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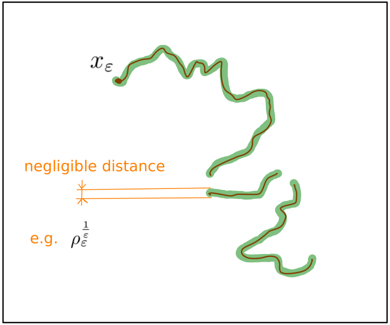

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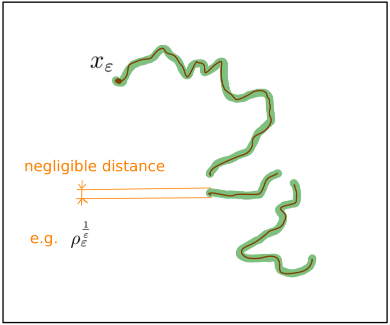

- Why arbitrary (ρ-moderate) representatives (x_ε)? It's necessary to have: intermediate value, mean value theorems...

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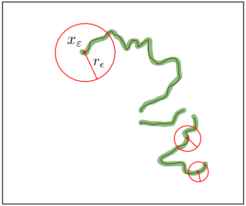

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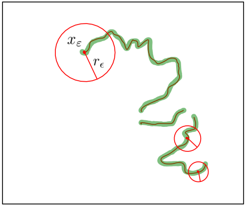

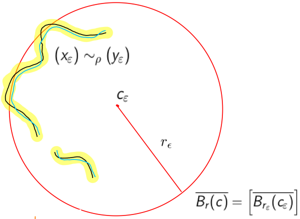

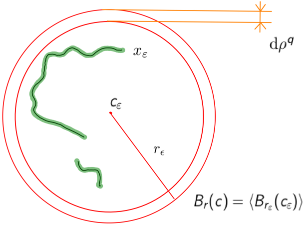

The universal way to have GF (i.e. with all the ρ-regularizations)

The idea: Set an infinitesimal $\sigma = [\sigma_\varepsilon]$ in a Gaussian density $N(x, \sigma)$: we get a net $(N(-, \sigma_\varepsilon))$ of smooth functions...

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- Universal property: any other way to get maps using nets of smooth functions is "smaller"

Grothendieck's rising sea method

- Schwartz's distributions are embedded and smooth functions preserved
- GSF are freely closed with respect to composition: $\delta\circ\delta$, $H\circ\delta$, $\delta\circ H$...
- One-dimensional integral calculus using primitives
- Classical theorems: intermediate value, (integral) mean value, Taylor's formulas, extreme value theorem, local and global inverse and implicit function theorems
- Multidimensional integration with generalized additivity and convergence theorems (pointwise convergence e.g. on $[a,b]^n$ implies uniform converge)
- ODE and PDE: Banach fixed point, Picard-Lindelöf for PDE, maximal set of existence, Gronwall, flux, Hadamard well-posedness, rel. with classical solutions...
- Calculus of variations and optimal control: Fundamental Lemma, second variation and minimizers, necessary Legendre condition, Jacobi fields, Conjugate points and Jacobi's theorem, Noether's theorem, Pontryagin
- Hyperfinite Fourier transform for all GSF, also of non tempered type
- Generalized holomorphic functions: we can extend compactly supported distributions from $\mathbb{R}$ to $\mathbb{C}$

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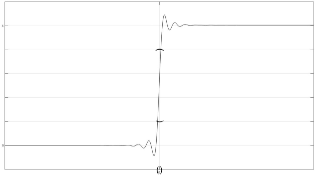
1. The sharp topology induces the discrete topology on $\mathbb{R}$ (but GSF are continuous)



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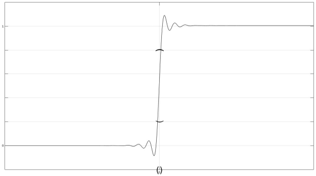

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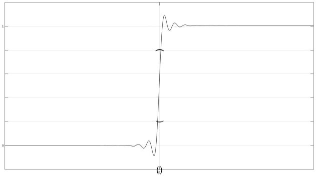

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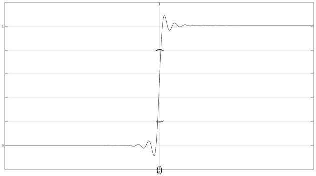

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3. Every non-Archimedean theory has similar "problems"

Physical modeling and GF (J. Marsden)

Actually, it rarely happens that the world is $\mathcal{C}^2$:

Length of the pendulum $\Lambda(\theta) = H(\theta_0 - \theta)L_1 + L_2$

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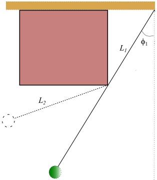

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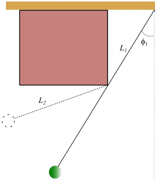

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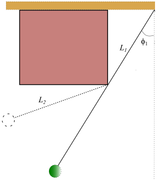

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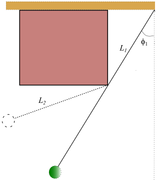

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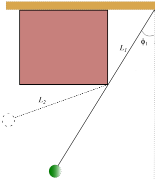

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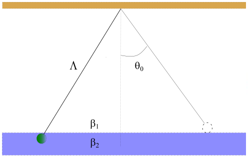
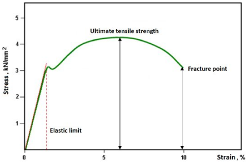
$$\ddot{\theta} + \dot{\theta}\frac{\dot{\Lambda}(\theta)}{\Lambda(\theta)} - g\frac{\dot{\Lambda}(\theta)}{\dot{\theta}\Lambda(\theta)^2}(\cos\theta - \cos\theta_0) + \frac{g}{\Lambda(\theta)}\sin\theta = 0$$

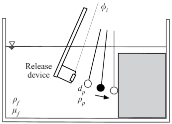
- Rigorous deduction of Snell's laws (discontinuous Lagrangians in optics)
- Finite and infinite potential wells in QM; formalization of uncertainty principle with Dirac delta
- Oscillatory motion of the pendulum in the interface of two media and non linear strain-stress model

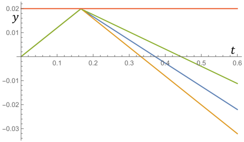




- Collision with coefficient of restitution $\frac{v_{\text{after}}}{v_{\text{before}}} \leq 1$

G. G. Joseph, R. Zenit, M. L. Hunt and A. M. Rosenwinkel
J. Fluid Mech. (2001), *vol.* 433, *pp.* 329–346.





Sheaf property for GSF: first remarks

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- Therefore, in order to have a beautiful sheaf property, it does not suffice to consider "sharply open set in ${}^\rho\widetilde{\mathbb{R}}^n$ (any $n \in \mathbb{N}$) and GSF", because we do not have a sheaf... in this way
- In the following ${}^\rho\mathcal{SGC}^\infty$ denotes the category of strongly internal sets and GSF

Definition

A subset K of ${}^{\rho}\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

- $K = [K_\varepsilon] := \left\{ [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid \forall^0 \varepsilon : \ x_\varepsilon \in K_\varepsilon \right\}$
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Sheaf property preliminaries: functionally compact sets

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3. $[-\mathrm{d}\rho^{-1}, \mathrm{d}\rho^{-1}]^n \supseteq \mathbb{R}^n$
4. Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$, we say X *admits a functionally compact exhaustion* if there exists a sequence $(K_q)_{q\in\mathbb{N}}$ such that $K_q \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$, $K_q \subseteq \mathrm{int}\,(K_{q+1})$, $X = \bigcup_{q\in\mathbb{N}} K_q$. For example, every strongly internal set $X = \langle A_\varepsilon \rangle$ (e.g. any ball) admits a functionally compact exhaustion

Sheaf property: the dynamic compatibility condition

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   1. $\forall\bar{\jmath} = (j_\varepsilon) \in [J]\,\forall[x_\varepsilon] \in U_{\bar{\jmath}} \cap K\,\forall\alpha \in \mathbb{N}^n: \ (\partial^\alpha f_{j_\varepsilon,\varepsilon}(x_\varepsilon)) \in \mathbb{R}^d_\rho$

   2. $\forall\bar{\jmath} = (j_\varepsilon), \bar{h} = (h_\varepsilon) \in [J]\,\forall[x_\varepsilon] \in K \cap U_{\bar{\jmath}} \cap U_{\bar{h}}: \ [f_{j_\varepsilon,\varepsilon}(x_\varepsilon)] = [f_{h_\varepsilon,\varepsilon}(x_\varepsilon)]$

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Finally, we say that $(f_j)_{j\in J}$ satisfies the DCC on the cover $(U_j)_{j\in J}$ if it satisfies the DCC on each functionally compact set contained in X

Sheaf property

Theorem

Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a set that admits a functionally compact exhaustion, $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{d}$ and $f \in \mathbf{Set}(X, Y)$. Let $X \subseteq \bigcup_{j \in J} U_j$, where for all $j \in J$ we have $f_j := f|_{U_j \cap X} \in {}^{\rho}\mathcal{GC}^{\infty}(U_j \cap X, Y)$ and U_j is a strongly internal set. Assume that $(f_j)_{j \in J}$ satisfies the dynamic compatibility condition on the cover $(U_j)_{j \in J}$. Then $f \in {}^{\rho}\mathcal{GC}^{\infty}(X, Y)$.

- The DCC implies the classical one
- The DCC is a necessary condition if we assume that the sections $(f_j)_{j \in J}$ glue together into a GSF
- Is this sheaf property based on the DCC a particular case of the general abstract notion of sheaf?

The category ${}^{\rho}\mathcal{G}\ell^{\infty}$ of glueable families 1/2

Definition

${}^{\rho}\mathcal{G}\ell^{\infty}$ is the category of *glueable families*, whose objects are non empty families $(U_j)_{j\in J} \in {}^{\rho}\mathcal{G}\ell^{\infty}$ of strongly internal sets in some space ${}^{\rho}\widetilde{\mathbb{R}}^u$:

$$J \neq \emptyset,\ \exists u \in \mathbb{N}\,\forall j \in J :\ {}^{\rho}\widetilde{\mathbb{R}}^u \supseteq U_j \in {}^{\rho}\mathcal{SGC}^{\infty}$$

We say that $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y}$ in ${}^{\rho}\mathcal{G}\ell^{\infty}$ if $\mathcal{X} = (U_j)_{j\in J}$, $\mathcal{Y} = (V_h)_{h\in H} \in {}^{\rho}\mathcal{G}\ell^{\infty}$ and $\varphi = \left((f_j)_{j\in J}, \alpha\right)$, where:

1. The map $\alpha \in \mathbf{Set}(J, H)$ is called a *reparametrization*
2. The family of GSF $f_j \in {}^{\rho}\mathcal{SGC}^{\infty}(U_j, V_{\alpha(j)})$, $j \in J$, satisfies the DCC on $U := \bigcup_{j\in J} U_j$
3. $U = \bigcup_{j\in J} U_j$ admits a functionally compact exhaustion

The category ${}^{\rho}\mathcal{G}\ell^{\infty}$ of glueable families 2/2

Composition and identities in ${}^{\rho}\mathcal{G}\ell^{\infty}$ are defined as follows: Let $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\psi} \mathcal{Z}$ in ${}^{\rho}\mathcal{G}\ell^{\infty}$, and set $\mathcal{X} = (U_j)_{j\in J}$, $\mathcal{Y} = (V_h)_{h\in H}$, $\mathcal{Z} = (W_l)_{l\in L}$, $\varphi = \left((f_j)_{j\in J}, \alpha\right)$, $\psi = \left((g_h)_{h\in H}, \beta\right)$. Then

$$U_j \xrightarrow{f_j} V_{\alpha(j)} \xrightarrow{g_{\alpha(j)}} W_{\beta(\alpha(j))} \quad \forall j \in J$$

$$J \xrightarrow{\alpha} H \xrightarrow{\beta} L,$$

and we hence set $\varphi \cdot \psi := \left(\left(f_j \cdot g_{\alpha(j)}\right)_{j\in J}, \alpha \cdot \beta\right)$, $1_{\mathcal{X}} := \left(\left(1_{U_j}\right)_{j\in J}, 1_J\right)$.

Lemma

${}^{\rho}\mathcal{G}\ell^{\infty}$ is a category and ${}^{\rho}\mathcal{SGC}^{\infty} \subseteq {}^{\rho}\mathcal{G}\ell^{\infty}$

Coverage of glueable functions

We now introduce a coverage on the category ${}^\rho\mathcal{G}\ell^\infty$ of glueable families:

Definition

Let $\mathcal{E} = (W_e)_{e\in E} \in {}^\rho\mathcal{G}\ell^\infty$. Then we say that $\gamma \in \Gamma(\mathcal{E})$ if there exists a non empty $J \in \mathbf{Set}$ such that:

1. $\gamma = (\gamma_j)_{j\in J}$ and $\gamma_j = \left((i_h)_{h\in J}, \delta\right)$ for all $j \in J$
2. $J \xrightarrow{\delta} E$ is a surjective map
3. $i_j : D_j \hookrightarrow W_{\delta(j)}$ for all $j \in J$, where $(D_j)_{j\in J} \in {}^\rho\mathcal{G}\ell^\infty$
4. $W_e = \bigcup \{D_j \mid \delta(j) = e,\ j \in J\}$ for all $e \in E$

Theorem

Γ is a coverage on ${}^\rho\mathcal{G}\ell^\infty$ and $({}^\rho\mathcal{G}\ell^\infty, \Gamma)$ is a concrete site

Definition

The category of sheaves ${}^\rho\mathrm{T}\mathcal{GC}^\infty := \mathrm{Sh}\left({}^\rho\mathcal{G}\ell^\infty, \Gamma\right)$ is called the *Grothendieck topos of generalized smooth functions*

The sheaf of glueable functions

The DCC is the key property to prove the following

Theorem

For each $\mathcal{Y} \in {}^{\rho}\mathcal{G}\ell^{\infty}$, *the functor* ${}^{\rho}\mathcal{G}\ell^{\infty}(-, \mathcal{Y})$ *is a concrete sheaf on the concrete site* $({}^{\rho}\mathcal{G}\ell^{\infty}, \Gamma)$: ${}^{\rho}\mathcal{G}\ell^{\infty}(-, \mathcal{Y}) \in {}^{\rho}T\mathcal{G}\mathcal{C}^{\infty}\,({}^{\rho}\mathcal{G}\ell^{\infty}, \Gamma)$.

FWF project proposal

Together with:

1. Katsuhiko Kuribayashi, Shinshu University
2. Kazuhisa Shimakawa, Okayama University
3. Norio Iwase, Kyushu University
4. Michael Kunzinger, University of Vienna

we submitted an interdisciplinary FWF project between the Department of Mathematics, Faculty of Science, Shinshu University and the Faculty of Mathematics, University of Vienna, where we propose to develop several ideas linking mathematical analysis and algebraic topology.

The main idea is to make differential homotopy theory with continuous functions embedded as GSF or synthetic (non-smooth) differential geometry with GSF

Diffeological spaces and generalized maps

- A diffeological space is a concrete sheaf on the concrete (Souriau) site of smooth functions. Generalized diffeological space := a concrete sheaf on the concrete site $({}^{\rho}\mathcal{G}\ell^{\infty}, \Gamma)$
- Find a characterization of concrete sheaves over $({}^{\rho}\mathcal{G}\ell^{\infty}, \Gamma)$ with the main aim of arriving at conditions as close as possible to the usual definition of diffeological space, but using strongly internal sets, GSF and their sheaf property (with DCC)
- Understand generalized maps between generalized diffeological spaces
- Prove that diffeological spaces are included in the topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$: Since the Souriau site $\mathcal{S} \subseteq {}^{\rho}\mathcal{SGC}^{\infty} \subseteq {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$, prove that also $\mathbf{Man} \subseteq \mathbf{Diff} \subseteq {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$

Extending diffeological spaces with non-Archimedean points

- Exactly as ${}^{\rho}\widetilde{\mathbb{R}}$ is obtained extending $\mathbb{R}$ by adding new non-Archimedean points, define an extension functor $X \in \mathbf{Diff} \mapsto \widetilde{X} \in {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$ and prove that $\widetilde{X}$ is a generalized diffeological space

- Why extension is important: $\delta|_{\mathbb{R}} = 0$, $(\delta \circ \delta)|_{\mathbb{R}} = \delta(0)$, $H|_{\mathbb{R}}$ does not satisfy the intermediate value theorem...

  Without considering non-Archimedean points we trivialize everything

- Ideas for extension functor**s**:
  1. $\widetilde{X} := \mathrm{colim}_{U \in \mathcal{S}/X} \langle U \rangle$, where $\langle U \rangle \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{u}$ is the strongly internal set generated by the open set $U \subseteq \mathbb{R}^{u}$ and the colimit is taken in ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$

  2. By observables: We say that a net $(x_{\varepsilon}) \in X^{I}$ is moderate (resp. negligible) if for all $f \in \mathbf{Diff}(U, \mathbb{R}^{d})$ such that $x_{\varepsilon} \in U \subseteq X$ for ε small, U open in the D-topology, we have $f \circ x$ is moderate (resp. negligible) in ${}^{\rho}\widetilde{\mathbb{R}}^{d}$

Preservation properties of these extension functors

- Study the preservation properties of the extension functor $X \in \mathbf{Diff} \mapsto \widetilde{X} \in {}^\rho\mathrm{T}\mathcal{GC}^\infty$:
  - counter-image and image: $\left(\widetilde{f}\right)^{-1}(\widetilde{U}) = \widetilde{f^{-1}(U)}$ and $\left(\widetilde{f}\right)(\widetilde{U}) = \widetilde{f(U)}$ for $f \in \mathbf{Diff}$
  - intersections and unions: $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$, $\widetilde{\bigcup_{i\in I} U_i} = \bigcup_{i\in I} \widetilde{U}_i$
  - complements: $\widetilde{X \setminus U} = \widetilde{X} \setminus \widetilde{U}$ or $\mathrm{int}\,\widetilde{(X \setminus U)} = \mathrm{int}\left(\widetilde{X} \setminus \widetilde{U}\right)$
  - subspaces, quotients and products
  - exponential objects
- Can we construct a standard part functor $(-)^\circ : {}^\rho\mathrm{T}\mathcal{GC}^\infty \longrightarrow \mathbf{Diff}$ at least for suitable near-standard subspaces (see e.g. Wu "The Fermat functors", TAC, 2016)?
  Recall: $\left({}^\rho\widetilde{\mathbb{R}}^n\right)^\bullet = \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists \lim_{\varepsilon\to 0^+} x_\varepsilon =: [x_\varepsilon]^\circ \in \mathbb{R}^n\}$ and not all generalized points have a standard part

Shimakawa homotopy theory with GSF

1. Can we realize Shimakawa's program of differential homotopy theory using GSF and ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$?

2. Prove that there exists a GSF diffeomorphism between the n-cube $[0,1]^n$, n-simplex Δ^n and n-ball B^n in ${}^{\rho}\widetilde{\mathbb{R}}^n$

3. We already obtain a GSF diffeomorphism $P : \overline{B_1(0)} \xrightarrow[\sim]{} [0,1]^2$ between the closed ball $\overline{B_1(0)} \subseteq {}^{\rho}\widetilde{\mathbb{R}}^2$ and the 2-cube defined by $[0,1] = \left\{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid 0 \le x \le 1\right\}$; moreover, $P|_{B_1^{\mathrm{E}}(0,0)} : \overline{B_1^{\mathrm{E}}(0)} \xrightarrow[\sim]{} [0,1]_{\mathbb{R}}^2$, where $[0,1]_{\mathbb{R}} := \{x \in \mathbb{R} \mid 0 \le x \le 1\}$

4. Generalize homotopy theory of diffeological spaces using GSF

Synthetic Differential Geometry

- Synthetic Differential Geometry (SDG) uses a non-Archimedean language of *nilpotent infinitesimals* ($h \neq 0$ but $h^n = 0$) to formalize the coordinate-free notions of differential geometry as originally developed by S. Lie, E. Cartan, H.G. Grassmann

- See Kock 1981 (2nd ed. 2006), Lavendhomme 1996, Moerdijk-Reyes 1990 (need topos model in intuitionistic logic), but also Giordano, Wu papers about "Fermat reals" in classical logic

- In SDG, one can synthetically consider tangent vectors and tangent modules, vector fields, existence and uniqueness of infinitesimal integral curves of a given vector field, Lie brackets, differential forms and their properties, infinitesimal Stokes theorem, de Rham currents, connections, parallel transport, curvature, etc. in smooth $M \in \mathbf{Man}$ but also in spaces of smooth mappings $\mathbf{Man}(N, M)$

Generalized Synthetic Differential Geometry

- We can introduce a language of nilpotent infinitesimals also in ${}^{\rho}\widetilde{\mathbb{R}}$: we say that $x =_j y$ (x *is equal to* y *up to* j*-th order infinitesimals*, $j \in \mathbb{R}_{>0}$) if

$$|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^{\frac{1}{j}})$$

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- If $\left(\forall^0 j \in \mathbb{R}_{>0} : \ x =_j y\right)$, then $x = y$

Generalized Synthetic Differential Geometry

- We can introduce a language of nilpotent infinitesimals also in ${}^{\rho}\widetilde{\mathbb{R}}$: we say that $x =_j y$ (x *is equal to* y *up to* j*-th order infinitesimals*, $j \in \mathbb{R}_{>0}$) if

$$|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^{\frac{1}{j}})$$

- If $k \in \mathbb{N}_{>0}$, we set $D_{kj} := \left\{h \in {}^{\rho}\widetilde{\mathbb{R}} \mid h^{k+1} =_j 0\right\}$, the *set of k-th order infinitesimals for the equality* $=_j$, and $D_{\infty j} := \left\{h \in {}^{\rho}\widetilde{\mathbb{R}} \mid \exists k \in \mathbb{N}_{>0} : \ h^{k+1} =_j 0\right\}$, the *set of infinitesimals for the equality* $=_j$

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- For all $j \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_{>0}$ there exist $e \in \mathbb{R}_{>0}$ such that $e \le j$, and $\forall h \in D_{ne} : \ f(x+h) =_j \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} h^r$. Using nilpotent infinitesimals every GSF is equals to its Taylor formula without remainder

Ideas for Generalized SDG

- A tangent vector at $x \in X$ for $=_j$ is an arrow of ${}^\rho\mathrm{T}\mathcal{GC}^\infty$ of the form $t : D_{1j} \longrightarrow X$ such that $t(0) = x$, intuitively an infinitesimal *generalized* linear line traced on X at x

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- Differential forms are generalized infinitesimal p-vectors (Grassmann's notion) at $x \in X$

Contacts and references

References:

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Thank you for your attention!