Reconstructing Schemes from their étale topoi

@ Toposes in Mondovì

Joint with. M. Carlson & S. Wolf

> Some results from work with C. Barwick & S. Glasman

Plan.

(1) Grothendieck's Conjecture

(2) Variants of conceptual completeness

(3) Speculation about relation to logic

(1) Grothendieck's Conjecture

Ntn. For a scheme X, write $X_{\text{ét}}$ for the étale topos of X.

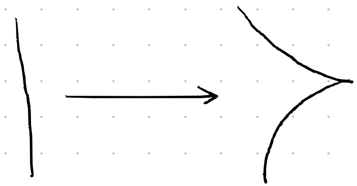
General Idea. To what extent is X determined by $X_{\text{ét}}$?

> For simplicity, we'll work with schemes over a field k.

Four "obvious" issues.

(1) Restriction to finite type morphisms: if $L \supset K$ is an extension of separably closed fields, then
$$\mathrm{Spec}(L)_{\text{ét}} \xrightarrow{\sim} \mathrm{Spec}(K)_{\text{ét}}$$

(2) Topological Invariance: If $f: X \longrightarrow Y$ is a universal homeomorphism, then
$$X \underset{Y}{\times} (-) : \text{Ét}_Y \longrightarrow \text{Ét}_X$$
is an equivalence of cats. Hence so is $f_*: X_{\text{ét}} \longrightarrow Y_{\text{ét}}$



- Hence need to invert universal homeomorphisms

(3) **Restriction to 'small' fields**: k alg closed, $\mathrm{char}(k)=0$
$X, Y / k$ smooth proper curves, then

$$X_{ét} \simeq Y_{ét} \text{ iff } g(X) = g(Y)$$

- Grothendieck: restrict to fields finitely generated over their prime fields.

(4) **Restriction on geometric morphisms**: If $f: X \longrightarrow Y$ is a morphism of schemes locally of finite type over a field k, then f sends closed points to closed points.

- This is not a topological property, and is not automatic for a geometric morphism $X_{ét} \longrightarrow Y_{ét}$ over $\mathrm{Spec}(k)_{ét}$.

Def. A geometric morphism $f_*: \mathcal{X} \longrightarrow \mathcal{Y}$ is **pinned** if the map on underlying spaces

$$|f_*|: |\mathcal{X}| \longrightarrow |\mathcal{Y}|$$

$|\mathcal{X}|$ = points of the locale $\mathrm{Sub}(1_{\mathcal{X}})$ of subterminal objects

sends closed points to closed points.

$|X_{ét}| = |X|$ underlying space

Prop (Carlson–H.–Wolf). S scheme, X, Y finite type S-schemes

Then the groupoid

$$\mathrm{Hom}_S^{\mathrm{pin}}(X_{ét}, Y_{ét})$$

of pinned geometric mors $X_{ét} \longrightarrow Y_{ét}$ over $S_{ét}$ is equivalent to a Set.

Conj (Grothendieck, 1983). If k is a finitely generated field, then the functor

$$\mathrm{Sch}_k^{ft}[\mathrm{UH}^{-1}] \longrightarrow \underbrace{\left(\begin{array}{c}\text{topoi over } \mathrm{Spec}(k)_{\text{ét}} \\ \text{and pinned geometric} \\ \text{morphisms}\end{array}\right)}_{(2,1)\text{-category}}$$

is fully faithful.

Thm (CHW). Grothendieck's conj. is true if k is infinite.

Reformulating the conjecture

> We can identify $\mathrm{Sch}_k^{ft}[UH^{-1}]$ explicitly.

Def. A ring A is:

(1) **Seminormal** if whenever $x^2 = y^3$ in A, $\exists!\ a \in A$ such that $x = a^3$ and $y = a^2$.

(2) **Absolutely weakly normal** if A is seminormal and for each prime ℓ and equation $\ell^\ell x = y^\ell$ in A, $\exists!\ a \in A$ st $x = a^\ell$ and $y = \ell a$

↳ true if ℓ is invertible in A: take $a = y/\ell$

Thm.

(1) The inclusion $\mathrm{Sch}^{awn} \hookrightarrow \mathrm{Sch}$ admits a right adjoint $X \mapsto X^{awn}$. Moreover

$$(-)^{awn} : \mathrm{Sch}[UH^{-1}] \xrightarrow{\sim} \mathrm{Sch}^{awn}$$

(2) A $\mathbb{Q}$-scheme X is awn iff X is seminormal

(3) An $\mathbb{F}_p$-scheme X is awn iff X is perfect, i.e., $\mathrm{Frob}: X \longrightarrow X$ is an iso.

Def. A k-scheme X is **topologically of finite type** if $X \longrightarrow \mathrm{Spec}(k)$ factors as

$$X \xrightarrow{UH} X' \xrightarrow{ft} \mathrm{Spec}(k).$$

Equivalent formulation of Grothendieck's conjecture.

k fg field, X and Y topologically of finite type $/k$ with X awn, then

$$\mathrm{Hom}_k(X,Y) \xrightarrow{\sim} \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}}).$$

> So our Thm generalizes:

Thm (Voevodsky, 1990). k fg field of char 0, $X, Y/k$ finite type with X normal. Then

$$\mathrm{Hom}_k(X,Y) \xrightarrow{\sim} \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

(2) Variants of conceptual completeness

Recall (Conceptual completeness, Makkai–Reyes).
If $\mathcal{X}$ and $\mathcal{Y}$ are coherent topoi, then TFAE for a geom mor $f_*: \mathcal{X} \longrightarrow \mathcal{Y}$:

(1) f_* is an equivalence

(2) f_* is coherent and $\mathrm{Pt}(f_*): \mathrm{Pt}(\mathcal{X}) \longrightarrow \mathrm{Pt}(\mathcal{Y})$ is an equivalence.

$f^*: \mathcal{Y} \to \mathcal{X}$ preserves coherent objects

Lem. Given qcqs schemes X and Y, if $|X|$ is a noetherian space, then every geometric morphism $X_{\acute{e}t} \longrightarrow Y_{\acute{e}t}$ is coherent.

Cor. For k an infinite finitely generated field and X and Y absolutely weakly normal and topologically of finite type over k, a morphism of k-schemes $f: X \longrightarrow Y$ is an isomorphism iff $\mathrm{Pt}(f_*): \mathrm{Pt}(X_{\acute{e}t}) \longrightarrow \mathrm{Pt}(Y_{\acute{e}t})$ is an equivalence of categories.

Hochster Duality & Strong Conceptual Completeness

Hochster's Stone Duality. There is an equivalence of categories

$$\left(\begin{array}{c}\text{Spectral Spaces}\\ \text{and quasicompact maps}\end{array}\right) \simeq \mathrm{Pro}(\mathrm{Poset}^{\mathrm{fin}})$$

Def (Barwick–Glasman–H.). A topos $\mathcal{X}$ is **spectral** if $\mathcal{X}$ is coherent and $\mathcal{C} = \mathrm{Pt}(\mathcal{X})$ has the following property:

(EI) For all $c \in \mathcal{C}$, every endomorphism $c \to c$ is an isomorphism.

Ex. If X is a qcqs scheme, then $X_{\text{ét}}$ is spectral.

Thm (BGH). There is an equivalence of (2,1)-cats

$$EI^{fin} \ni \mathcal{C} \longmapsto \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$$

$$\begin{tikzcd} \mathrm{Pro}(EI^{fin}) \arrow[r, shift left, "\sim"] \arrow[dr, "\lim"'] & \left(\begin{array}{c}\text{Spectral topoi}\\ \text{and coherent}\\ \text{geometric mors}\end{array}\right) =: \mathrm{RTop}^{Spec} \arrow[l, shift left, "\sim"', "\hat{\Pi}_{(\infty,1)}"] \arrow[d, "Pt"] \\ & \mathrm{Cat} \end{tikzcd}$$

Moreover, for any Spectral topos $\mathcal{X}$ with corresponding profinite EI category $\hat{\Pi}_{(\infty,1)}(\mathcal{X}) = \{\Pi_i\}_{i \in I}$,

$$\mathcal{X}^{coh} \simeq \mathrm{Fun}^{cts}(\hat{\Pi}_{(\infty,1)}(\mathcal{X}), \mathrm{Set}^{fin})$$

$$\stackrel{def}{=} \operatorname*{colim}_i \mathrm{Fun}(\Pi_i, \mathrm{Set}^{fin}).$$

Ntn. $B : \mathrm{Cat}_\infty \longrightarrow \mathrm{Gpd}_\infty$ left adjoint to the inclusion "classifying space"

$$\rightsquigarrow B : \mathrm{Pro}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Pro}(\mathrm{Gpd}_\infty)$$
$$\{e_i\}_{i\in I} \longmapsto \{Be_i\}_{i\in I}$$

Thm (BGH). For X spectral, $B\hat{\Pi}_{(\infty,1)}(X)$ recovers Lurie's shape $\Pi_\infty(X)$.

Cor. For k an infinite fg field, the functor

$$\mathrm{Sch}_k^{\mathrm{awn,tft}} \xrightarrow{\mathrm{Gal}} \mathrm{Pro}(\mathrm{EI}^{\mathrm{fin}})^{\mathrm{pin}}$$
$$X \longmapsto \mathrm{Gal}(X) := \hat{\Pi}_{(\infty,1)}(X_{\acute{e}t})$$

is fully faithful.

Condensed categories of points

Thm (BGH). The functor

$$\underline{\mathrm{Pt}}^{\mathrm{coh}} \quad \mathrm{RTop}^{\mathrm{Spec}} \longrightarrow \mathrm{Cond}(\mathrm{Cat})$$

$$X \longmapsto [K \mapsto \underset{\substack{\shortparallel \\ \left(\text{coherent geometric mors } f^*: X \to \mathrm{Sh}(K)\right)}}{\mathrm{Fun}^*_{\mathrm{coh}}}(X, \mathrm{Sh}(K))]$$

is fully faithful.

Thm (Lurie, after Makkai). The functor

$$\underline{\mathrm{Pt}}: \begin{pmatrix} \text{coherent topoi} \\ \text{and all} \\ \text{geometric mors} \end{pmatrix} \longrightarrow \mathrm{Cond}(\mathrm{Cat})$$

$$X \longmapsto [K \mapsto \overset{\substack{\left(\text{all geometric mors } f^*: X \to \mathrm{Sh}(K)\right) \\ \shortparallel}}{\mathrm{Fun}^*}(X, \mathrm{Sh}(K))]$$

is fully faithful.

(3) Speculation about relation to logic

Ntn. $\mathbb{T}$ complete first order theory

$\rightsquigarrow$ Lascar group $\mathrm{Gal}_L(\mathbb{T})$ topological group depending on a sufficiently saturated model $\mathcal{U}$

› $\mathrm{Mod}_{\mathbb{T}} \simeq \mathrm{Pt}(\underbrace{\mathrm{Set}[\mathbb{T}]}_{\text{classifying topos}})$

Thm (Campion - Cousins - Ye). As a discrete group,

$$\mathrm{Gal}_L(\mathbb{T}) \cong \pi_1(B\mathrm{Mod}_{\mathbb{T}}, \mathcal{U}).$$

Idea. Condensed math can make this a topological iso.

Ntn. $B^{cond}: \mathrm{Cond}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Cond}(\mathrm{Gpd}_\infty)$

$$\mathcal{C} \longmapsto [K \mapsto B(\mathcal{C}(K))]$$

left adjoint to inclusion.

Q. $\pi_1^{cond}(B^{cond}\,\underline{\mathrm{Pt}}(\mathrm{Set}[\mathbb{T}])) \cong \mathrm{Gal}_L(\mathbb{T})$?

Remark (work in progress with many). For a qcqs scheme X, the condensed group $\pi_1^{cond}(B^{cond}\,\underline{\mathrm{Pt}}^{coh}(X_{ét}))$ is a refinement of $\pi_1^{proét}(X)$ defined by Bhatt-Scholze.

proétale fundamental group