The colimit of all monomorphisms classifies hyperconnected geometric morphisms

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Based on [Internal parameterization of hyperconnected quotients, Hora 2024]

Overview

Main Definition (LSC)

The local state classifier (**LSC**) of a category $\mathcal{C}$ is the colimit of all monomorphisms of $\mathcal{C}$, if it exists.

Main Theorem (Classification of hyperconnected geometric morphisms)

Every Grothendieck topos $\mathcal{E}$ has LSC Ξ, and hyperconnected geometric morphisms from $\mathcal{E}$ are in one-to-one correspondence with semilattice homomorphisms $\Xi \to \Omega$.

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Background (1/2)

In his [open problems in topos theory, Lawvere 2009], William Lawvere asks:

Lawvere's open problems #1 (Quotient toposes)

[...] Is there a Grothendieck topos for which the number of these **quotients** is not small? At the other extreme, could they be **parameterized internally, as subtoposes are?**

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- [Kamio and Hora 2024] proves the topos $\mathbf{PSh}(M_\omega)$ has a non-small number of quotients.
- This talk is about [Hora 2024] internal parameterization of **hyper**connected geometric morphisms.

Background (2/2)

"External" ←— bijection —→ "Internal"

Subtopos	←— by Lawvere and Tierney —→	Lawvere Tierney topology $\Omega \to \Omega$	Known
Connected g.m.	←— Lawvere's open problem —→	✗	[Y.Kamio, H. 2024]
$\cup$			
Hyperconnected g.m.	←— Main Theorem —→	Main Definition $\Xi \to \Omega$	This Talk

Ξ: local state classifier

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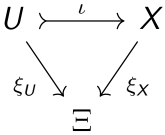
Local state classifier

Definition (LSC: local state classifier)

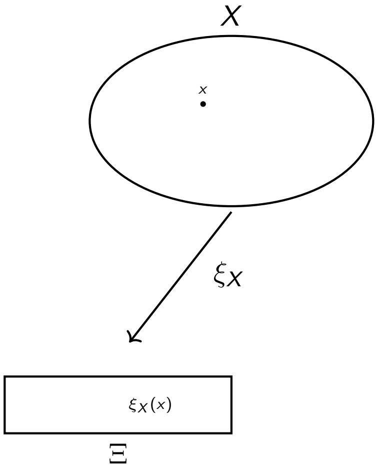
A local state classifier Ξ of a category $\mathcal{C}$ is a colimit of all monomorphisms:

$$\Xi = \operatorname{colim}(\mathcal{C}_{\text{mono}} \rightarrowtail \mathcal{C}).$$

The associated cocone is referred to as $\{\xi_X : X \to \Xi\}_{X \in \mathrm{ob}\mathcal{C}}$.



ξ_X is "locally determined"



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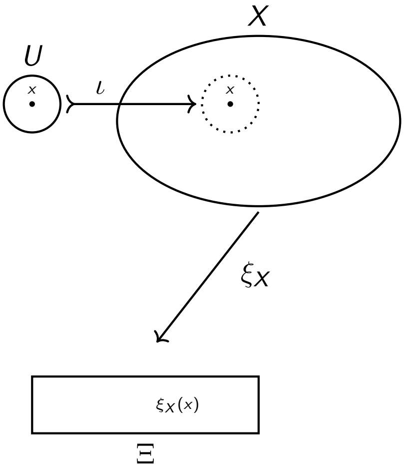

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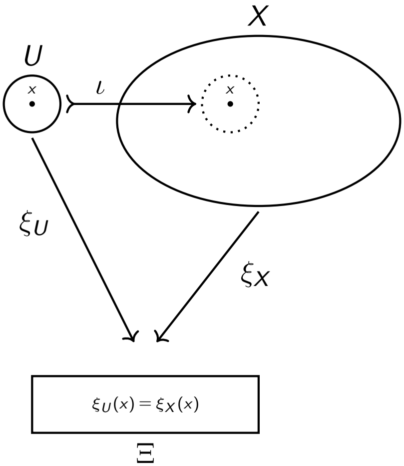

Exsistence

Proposition (Existence)

Every Grothendieck topos has a LSC.

In particular, the LSC of presheaf topos has a good description: **"dual" to the subobject classifier!**

Proposition (LSC of presheaf topos)

For a small category $\mathcal{C}$, the LSC Ξ of $\mathbf{PSh}(\mathcal{C})$ is the presheaf of quotient objects of representables.

$$\Xi(c) = \{\mathrm{y}(c) \twoheadrightarrow Q\}$$

Internal Semilattice structure

Proposition (Semilattice structure)

If a cartesian closed category (in particular, a topos) $\mathcal{C}$ has a LSC Ξ, then Ξ admits a canonical internal semilattice structure.

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle \xi_X \times \xi_Y} \swarrow & & \searrow {\scriptstyle \xi_{X \times Y}} \\ \Xi \times \Xi & \xrightarrow{\quad \wedge \quad} & \Xi \end{array}$$

So far, we've seen

1. A LSC of a category $\mathcal{C}$ is a colimit of all monomorphisms.
2. Every Grothendieck topos has a LSC.
3. If a cartesian closed category has a LSC Ξ, then Ξ has a semilattice structure.

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Hyperconnected geometric morphism

Definition (Hyperconnected geometric morphism [Johnstone 1981])

A geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ is **hyperconnected** if f^* is fully faithful, and its counit is monic.

Example (Topological groups)

For a topological group G, the inclusion $\mathbf{Cont}(G) \hookrightarrow \mathbf{PSh}(G^\delta)$ defines a hyperconnected geometric morphism $\mathbf{PSh}(G^\delta) \to \mathbf{Cont}(G)$

(See [Johnstone 1981] and [Rosenthal 1982])

Main Theorem

Theorem (Main Theorem)

If a topos $\mathcal{E}$ has a LSC Ξ, there is a one-to-one correspondence between

- *hyperconnected geometric morphisms from $\mathcal{E}$,*
- *internal semilattice homomorphisms $\Xi \to \Omega$,*
- *(internal filters $F \rightarrowtail \Xi$)*

A sketch of the correspondence:

1. A hyperconnected geometric morphism $f : \mathcal{E} \to \mathcal{F}$.
2. The monic counit $\{\epsilon_X : f^* f_*(X) \rightarrowtail X\}_{X \in \mathrm{ob}(\mathcal{C})}$.
3. A "good" family of morphisms $\{X \to \Omega\}_{X \in \mathrm{ob}(\mathcal{C})}$.
4. A semilattice homomorphism $\Xi \to \Omega$.

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Degenerate example

Example (Set)

The LSC of **Set** is a singleton. In fact, there are no non-trivial hyperconnected geometric morphisms from **Set**.

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Theorem (Characterization of localic topoi)

A Grothendieck topos $\mathcal{E}$ is localic if and only if its LSC Ξ is terminal.

Toy example: Graphs

Example (Directed Graphs)

The LSC of the topos of directed graphs $\mathrm{DirGraph} = \mathbf{PSh}(\cdot \rightrightarrows \cdot)$ is given by the "2-bouquet":

There are two hyperconnected geometric morphisms corresponding to

and

Group actions

Let G be a group regarded as a one-object category.

Example (Group action topos)

The LSC of the topos of G-sets $\mathbf{PSh}(G)$ is the set of all subgroups:

$$\Xi = \mathrm{Sub}_{\mathrm{Group}}(G).$$

ξ_X sends each element $x \in X$ to its **stabilizer subgroup**.

From this observation, we can prove that every (hyper)connected geometric morphisms from $\mathbf{PSh}(G)$ is induced by a topology on G.

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Existence and boundedness

- Every Grothendieck topos has a LSC.
- For a finite category $\mathcal{C}$, the finite presheaf topos $\mathbf{finPSh}(\mathcal{C})$ has a LSC.

Example

$\mathbf{FinSet}^{\mathbb{Z}}$ does not have a LSC.

Question

When does a topos have a LSC? Is it related to boundedness?

Preservation by functor

To study some categorical structure S, it is usual to study a class of functors that preserve the structure S.

Question

What kind of functors preserve LSC?

Even a sheafification functor might not preserve LSC:

Example

The sheafification functor $\mathrm{Edge}\colon \mathbf{PSh}(\cdot \rightrightarrows \cdot) \to \mathbf{Set}$ does not preserve LSC.

A generalization of LSC

Question

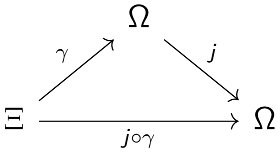
To what extent can these discussions be generalized?

LSC should be generalized to broader contexts, like

- **Enriched** LSC in enriched contexts (for classifying subcategories of modules?).
- **Not mono** LSC relative to factorization systems (not only $(\mathrm{Epi}, \mathrm{Mono})$).
- **Not cartesian** Monoid structure on LSC w.r.t. (non-cartesian) monoidal structure.
- **Internal** In the internal language of a topos.

Relation to subtopoi

From a hyperconnected geometric morphism ($\leftrightarrow \gamma$) and a subtopos ($\leftrightarrow j$, we can construct a hyperconnected geometric morphism ($\leftrightarrow j \circ \gamma$).



Question

What is this interaction between hyperconnected geometric morphisms and subtopoi?

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