Projective Space and Line Bundles in Synthetic Algebraic Geometry

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Toposes in Mondovì, 2024

In this talk:

- SAG at a glance
- projective space $\mathbb{P}^n$
- line bundles, $\mathrm{Pic}(X)$
- classification of line bundles on $\mathbb{P}^n$
- application to $\mathrm{Aut}(\mathbb{P}^n)$

All results are well-known in (external) algebraic geometry, but we present new, synthetic proofs using higher types.

Synthetic algebraic geometry

$$k\text{-}\mathrm{Sch}_{\mathrm{f.p.}} \quad \hookrightarrow \quad \mathrm{Zar}_k := \mathrm{Sh}(k\text{-}\mathrm{Alg}_{\mathrm{f.p.}}{}^{\mathrm{op}}, J_{\mathrm{Zar}})$$

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We interpret HoTT internally in $\mathrm{Zar}_k^{(\infty,1)}$ and write R for the *structure sheaf*:

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- R is a ring.
- Every $x : R$ with $x \neq 0$ is invertible.
  - But we don't have $x = 0 \vee x \neq 0$.
- Every $x : R^n$ with $x \neq 0$ generates a sub-module $\langle x \rangle \subseteq R^n$ with $\langle x \rangle \cong R^1$.

- Every function $f : R \to R$ is a polynomial.
  - But we can't determine $\deg(f) : \mathbb{N}$.
- Every function $R^m \to R^n$ is given by n polynomials in m variables.

Key features of HoTT (homotopy type theory)

- A type X can be a proposition, set, groupoid, 2-groupoid, ...
- We can form the *truncations* $\|X\| = \|X\|_{\mathrm{prop}}$, $\|X\|_{\mathrm{set}}$, $\|X\|_{\mathrm{grpd}}$, ...
- $\prod_{x:X} Y(x)$ generalizes "$\forall x \in X.\, Y(x)$".
- $\sum_{x:X} Y(x)$ generalizes "$\{\, x \in X \mid Y(x) \,\}$".
- Isomorphisms $X \cong Y$ are the same as identifications $X = Y$.

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$= \mathbb{G}_1(R^{n+1})$

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Interpolating between two points in $\mathbb{P}^n$

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So we have

$$\mathbb{G}_1(\langle p, p' \rangle) \subseteq \mathbb{P}^n$$

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We say: $\mathbb{G}_1(\langle p, p' \rangle)$ is the "line" interpolating between p and p'.

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Fix $p_0 \neq p_1$. Then:

$$\begin{array}{ccc} p \neq p_0 & \vee & p \neq p_1 \\ f(p) = f(p_0) & \vee & f(p) = f(p_1) = f(p_0) \end{array}$$

Recap of linear algebra (tensor product)

The following is true for any ring R.

$$- \otimes - : R\text{-}\mathrm{Mod} \times R\text{-}\mathrm{Mod} \to R\text{-}\mathrm{Mod}$$

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$$M \otimes M^\vee \to R^1$$

$$R^1 \otimes {R^1}^\vee \xrightarrow{\sim} R^1$$

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Definition
The *Picard group* of X is

$$\mathrm{Pic}(X) :\equiv \| X \to BR^\times \|_{\mathsf{set}}.$$

Line bundles on $\mathbb{P}^n$

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Recall: $\mathbb{P}^n := \sum_{L \subseteq R^{n+1} \text{ sub-module}} \|L \cong R^1\|$

The *tautological line bundle* on $\mathbb{P}^n$ is:

$$\begin{aligned} \mathcal{O}(-1) : \mathbb{P}^n &\to BR^\times \\ L &\mapsto L \end{aligned}$$

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Q: Are there other line bundles on $\mathbb{P}^n$?

$\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$

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Theorem

For every line bundle $L : \mathbb{P}^n \to BR^\times$ there is a number $d : \mathbb{Z}$ such that $\|L = \mathcal{O}(d)\|$. Thus:

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Plan for $n \geq 2$:

- Strengthen the $n = 1$ case to a non-truncated statement.
- Adjust L so that we can expect $\|L = \mathcal{O}(0)\|$.
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Corollary: If $\deg(L) = 0$ then we have $\prod_{p,p' : \mathbb{P}^1} L(p) = L(p')$.

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Thus: $L(p) = L(p')$ for all p, p' on $\mathbb{G}_1(P)$.

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internally in Zar

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For $p \neq p'$ in $\mathbb{P}^n$ we have $\deg(L|_{\mathbb{G}_1(\langle p, p' \rangle)}) = 0$, so:

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Fix standard points $p_0, p_1 : \mathbb{P}^n$.

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For $p : \mathbb{P}^n \setminus \{p_0, p_1\}$ we have *two* identifications:

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Fact: Every function $\mathbb{P}^n \setminus \{p_0, p_1\} \to R^\times$ is constant.

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For $p \neq p'$ in $\mathbb{P}^n$ we have $\deg(L|_{\mathbb{G}_1(\langle p,p' \rangle)}) = 0$, so:

$$p \neq p' \quad \to \quad L(p) = L(p')$$

Fix standard points $p_0, p_1 : \mathbb{P}^n$ and paths $L(p_0) = R^1$, $L(p_1) = R^1$.

$L = \text{const}\, R^1 \quad \text{on } \mathbb{P}^n \setminus \{p_0\}$
$L = \text{const}\, R^1 \quad \text{on } \mathbb{P}^n \setminus \{p_1\}$

For $p : \mathbb{P}^n \setminus \{p_0, p_1\}$ we have *two* identifications:

$$R^1 = L(p) = R^1$$

Fact: Every function $\mathbb{P}^n \setminus \{p_0, p_1\} \to R^\times$ is constant.

So we conclude: $L = \mathcal{O}(0)$.

Application: $Aut(\mathbb{P}^n)$

internally in Zar

Recall: Every function $R^m \to R^n$ is given by n polynomials in m variables.

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Corollary

Every automorphism $\mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ is given by an invertible matrix, unique up to scalar multiplication.

$$\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(R)$$