Some Applications of Toposes of Measure-Theoretic Sheaves

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Toposes in Mondovì

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A Simple Example

Theorem (Extreme Value Theorem)

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A compact interval $[a, b]$ is the historically first special case a proof of which relies on two main ingredients: the Bolzano–Weierstraß theorem and the Dedekind completeness of the real numbers.

Randomization

- Let $(X, \mathcal{F}, \mu)$ be a probability space, let $K \colon X \rightrightarrows \mathbb{R}$ be a compact-valued correspondence, and let $f \colon X \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(x, y)$ is continuous in y for every x.

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for every open set O in $\mathbb{R}$.
- Additionally, if f is a Carathédory integrand, that is, $f(x, y)$ is measurable in x for every y, then the optimal value function is measurable.

Measurable Selection Theory

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- These assumptions cannot essentially be relaxed if we replace pointwise everywhere formalization with an almost everywhere one.
- Applications include stochastic programming (e.g., portfolio optimization), optimal control theory (e.g., differential inclusions), calculus of variations (e.g., energy minimization), game theory (e.g., existence of Nash equilibrium), and mathematical economics (e.g., decision theory under uncertainty).

A Direct Point-Free Approach

- Let $L^0 = L^0(X \to \mathbb{R})$ denote the space of measurable functions $f\colon X \to \mathbb{R}$, where two functions f and g are identified if $\mu(\{x \in X\colon f(x) \neq g(x)\}) = 0$.

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- There are essentially two choices for convergence on L^0: the topology of convergence in probability, which is metrizable (and a ring topology), or almost sure convergence, which is generally not even topologizable.
- Let $f, g \in L^0$ with $f \leq g$ almost surely. Consider the random (point-free) interval

$$K = [f, g] = \{h \in L^0 : f \leq h \leq g\}.$$

  In general, K is not compact in the topology of convergence in probability, though it is closed under almost sure convergence, and bounded with respect to the almost sure ordering.

A measurable Euclidean Topology

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- Notice that an open interval (f, g) satisfies the following concatenation property: given a countable measurable partition (A_n) of X and $h_n \in (f, g)$, the function h defined by $h = h_n$ on A_n is an element of (f, g) as well.

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- A collection $\mathcal{C}$ of subsets of L^0 is said to 'cover' a set X if, for any element $f \in X$, there exists a partition (A_n) and elements $f_n \in Y_n$ for some $Y_n \in \mathcal{C}$ such that $f = f_n$ on A_n for all n.

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- A set X in L^0 is 'compact' if, for every covering $\mathcal{C}$ consisting of open intervals, there exists a partition (A_n) and for each n a finite subcollection $\mathcal{C}_n \subset \mathcal{C}$ such that for every $f \in X$, there is an $f_n \in Y_n \in \mathcal{C}_n$ with $f = f_n$ on A_n for all n.

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- The interval $[f, g]$ satisfies this notion of compactness.

A Measurable Extreme Value Theorem

Theorem

Let $F\colon [f,g] \to L^0$ be continuous in the interval topology and respect concatenations in the sense that if $h \in [f,g]$ is such that $h = h_n$ on A_n for some partition (A_n) and $h_n \in [f,g]$, then $F(h) = F(h_n)$ on A_n.
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- L^0 is Dedekind complete with respect to the almost sure order.
- We also have a Bolzano–Weierstraß-type theorem: Let (f_n) be a bounded sequence. Then there exists an increasing sequence of random indices $N_1(x) < N_2(x) < \dots$ and an $f \in L^0$ such that $f_{N_k(x)}(x) \to f(x)$ almost surely.

A Generalization of Measurable Selection Techniques

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- There is a one-to-one relationship between Carathéodory integrands and continuous functions from L^0 to itself in the interval topology, respecting countable concatenations.
- The direct procedure we outlined can be generalized to allow values in inseparable spaces (the modified notion of compactness is flexible), and we can replace closed-valued Effros measurable correspondences with any subset of L^0 that is closed under countable concatenations.

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- In fact, we adopt a point-free and direct approach that bypasses fiberwise applications of classical theorems and measurable selection theory. Instead, we identify objects, relations, and concepts that admit provable measurable versions of classical theorems. The application of these measurable versions leads directly to measurable outcomes.
- The underlying assumptions are that the measure space has σ-finite measure, and all objects and relations are identified up to almost sure equivalence, and respect countable concatenations.

Toposes of Measure-theoretic Sheaves

- Identifying at the primary level of the probability space $(X, \mathcal{F}, \mu)$ leads to the associated measure algebra $\mathcal{A} = \mathcal{F}/\mathcal{N}$, where $\mathcal{N}$ is the σ-ideal of null sets. This quotient algebra satisfies two key properties:

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- This topos has a rich internal logic: it has a natural numbers object, it is Boolean, and satisfies the axiom of choice.
- In fact, one can show that the objects of this topos form a Boolean-valued model of ZFC. We have just seen an example of the internal discourse of this topos.

The Real Numbers in $\mathfrak{Sh}$

- For $a \in \mathcal{A}$, let $L^0(a)$ denote the space of restrictions of measurable functions to a representative of a, and for $a \leq b$ in $\mathcal{A}$, let $\phi_{a,b} : L^0(b) \to L^0(a)$ be the restriction map. Then $\mathbb{R}_{\mathfrak{Sh}} = \{L^0(a)_{a \in \mathcal{A}}, (\phi_{a,b})_{a,b \in \mathcal{A},\, a \leq b}\}$ is an object of $\mathfrak{Sh}$ - in fact, it is (up to isomorphy) the real numbers of $\mathfrak{Sh}$.

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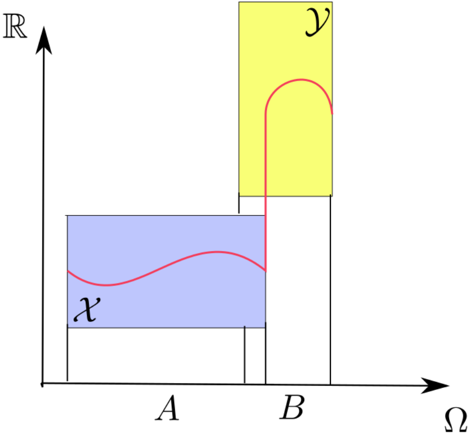
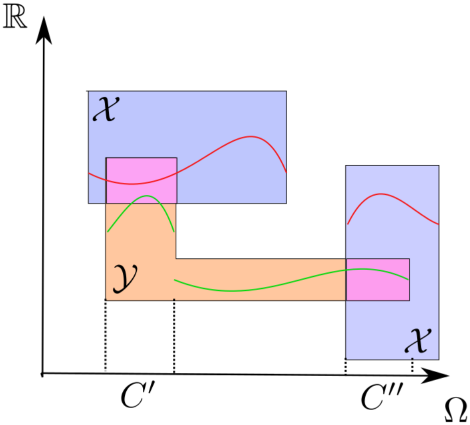
- The measurable version of the Bolzano–Weierstraß theorem is an external interpretation of the Bolzano–Weierstraß theorem inside $\mathfrak{Sh}$.
- The power set of $\mathbb{R}_{\mathfrak{Sh}}$ can be described as pairs (X, a) where $a \in \mathcal{A}$ and $X \subset L^0$ is closed under countable concatenations.

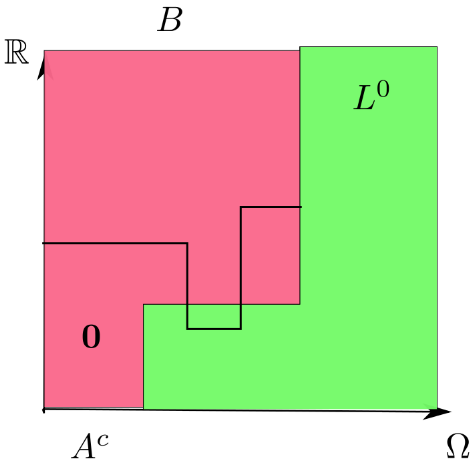
Set union



Set intersection



Set complement



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- More abstractly, for any σ-complete Boolean algebra $\mathcal{E}$, let $\mathrm{Hom}(\mathcal{E}, \mathcal{A})$ denote the collection of Boolean σ-homomorphisms. In general, we only have the inclusion $L^0(X \to E) \subset \mathrm{Hom}(\mathcal{E} \to \mathcal{A})$ if $\mathcal{E}$ is an algebra of subsets of some set E.

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- Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. We define

$$L^2(\mathcal{F}|\mathcal{G}) = \{f \in L^0 : \mathbb{E}(|f|^2|\mathcal{G}) < \infty\}$$

  This is a subset of $L^0(\mathcal{F})$ closed under countable concatenations. It is a $\mathfrak{Sh}$ (complex) Hilbert space with $\mathfrak{Sh}$-inner product

$$\langle f, g \rangle := \mathbb{E}(f \cdot g|\mathcal{G})$$

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Two Applications to Ergodic Structure Theory

- Fix an arbitrary discrete group Γ.
- Let $(\mathcal{A}, \mu)$ be a probability algebra and let $\mathrm{Aut}(\mathcal{A}, \mu)$ denote the group of measure-preserving Boolean automorphisms of $(\mathcal{A}, \mu)$.
- We define a measure-preserving dynamical system (MPDS) to be a tuple $(\mathcal{A}, \mu, T)$, where $(\mathcal{A}, \mu)$ is a probability algebra and $T\colon \Gamma \to \mathrm{Aut}(\mathcal{A}, \mu)$ is a group homomorphism.

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- A factor of an MPDS $(\mathcal{A}, \mu, T)$ is another MPDS $(\mathcal{B}, \nu, S)$ such that there exists a measure-preserving Boolean homomorphism $\pi \colon \mathcal{B} \to \mathcal{A}$ that intertwines the action:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\pi} & \mathcal{A} \\ \downarrow S^{\gamma} & & \downarrow T^{\gamma} \\ \mathcal{B} & \xrightarrow{\pi} & \mathcal{A} \end{array}$$

Koopman Representation

- The Koopman unitary representation of Γ on $L^2(\mathcal{A}, \mu)$ splits as:

$$L^2(\mathcal{A}, \mu) = \mathrm{AP} \oplus \mathrm{WM},$$

where AP corresponds to the finite-dimensional representations (the almost periodic/compact/structured part of the system $(\mathcal{A}, \mu, T)$), and WM is the part that does not admit any finite-dimensional representation (the weakly mixing/random part of the system).

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- This primary decomposition, however, does not capture the fine structure of $(\mathcal{A},\mu,T)$. For example, consider the $\mathbb{Z}$-system $\mathbb{T}\times\mathbb{T}$ where the dynamics are induced by:

$$(x,y) \mapsto (x+\alpha, y+x).$$

  The almost periodic part of this system is the factor $\mathbb{T}$, with $x \mapsto x+\alpha$. However, relative to this factor, the whole system $\mathbb{T}\times\mathbb{T}$ also behaves almost periodically. The global system $\mathbb{T}\times\mathbb{T}$ is a compact extension of its compact factor $\mathbb{T}$.

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- Classically, this is done using measurable Hilbert space bundles and direct integrals using the disintegration of measure of the factor map $\pi\colon \mathcal{B} \to \mathcal{A}$.
- This approach is similar to the one outlined in the previous optimal value problem and suffers from the same restrictions. Indeed, measure disintegrations only exist under countability assumptions on the acting group and/or separability assumptions on the probability algebras.
- Alternatively, we can translate the problem into the topos of measure-theoretic sheaves on the factor algebra. In this context, the representation theory of the compact extension can be established by the internal Hilbert space theory of the sheaf $L^2(\mathcal{F}|\mathcal{G})$. This removes countability and separability restrictions. In fact, the representation theory of compact extensions follows from the logical transfer principle (J., ETDS 2023).

Measurable Cohomology

- Assume Γ is abelian. Let (X, μ, T) be a Γ-system, and let K be a compact abelian group. A cocycle is a map $\rho \colon \Gamma \times X \to K$ satisfying for all $\gamma_1, \gamma_2 \in \Gamma$ the cocycle identity

$$\rho_{\gamma_1+\gamma_2} = \rho_{\gamma_1} + \rho_{\gamma_2} \circ T^{\gamma_1}.$$

  A cocycle ρ is a coboundary if there exists a measurable map $F \colon X \to K$ such that for all $\gamma \in \Gamma$,

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- A pointwise formalization of this theorem leads to restrictions, such as Γ being countable, K being metrizable, and (X, μ) being separable.
- Following a point-free formalization, Tao and I (ETDS, 2023) proved a more general version of the Moore–Schmidt theorem using the internal Pontryagin duality between compact and discrete abelian groups in the topos over the measure algebra of (X, μ).

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- This interpretation suggests a way to understand vector duality when E^* does not satisfy the Radon–Nikodým property (work in progress).

Thanks for listening!