What is the spectrum of quantum algebra?

(twilight zone between local and non-local)

Maxim Kontsevich
IHES, France

Theorem (D.Radchenko, M.Viazovska, 2017): the linear map from the Schwartz space $S(\mathbb{R})$ to the space of functions on $\mathfrak{W} =$ disjoint union of two copies of $\{0, \pm\sqrt{1}, \pm\sqrt{2}, \dots\}$, given by

$$f \mapsto \begin{cases} (\dots, f(-\sqrt{2}), f(-\sqrt{1}), f(0), f(\sqrt{1}), f(\sqrt{2}), \dots) \\ (\dots, \widehat{f}(-\sqrt{2}), \widehat{f}(-\sqrt{1}), \widehat{f}(0), \widehat{f}(-\sqrt{1}), \widehat{f}(\sqrt{2}), \dots), \end{cases}$$

where $f(\xi) = \int_{-\infty}^{\infty} e^{-2\pi \mathrm{i}\,\xi x} f(x) dx$ is the Fourier transform of f, identifies $S(\mathbb{R})$ with the codimension 1 subspace in the space of faster than polynomially decaying functions on $\mathfrak{W}$, cut out by the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,.$$

Rough generalization: values $f(\pm n^{\alpha_1})_{n=0,1,2,\dots}$ and $\widehat{f}(\pm n^{\alpha_2})_{n=0,1,2,\dots}$ where $0 < \alpha_1, \alpha_2 < 1$, define $f \in S(\mathbb{R})$ uniquely if $\alpha_1 + \alpha_2 < 1$ and non-uniquely if $\alpha_1 + \alpha_2 > 1$.

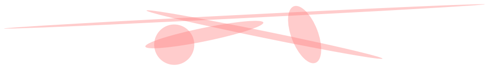
Easy classical fact: the linear map from the Schwartz space $S(\mathbb{R})$ to the space of functions on $\mathbb{Z}^2$ given by

$$f \mapsto \Big(\int_{-\infty}^{\infty} e^{-\pi(x+n)^2+2\pi\mathrm{i}\,mx} f(x)dx\Big)_{(m,n)\in\mathbb{Z}^2}$$

identifies $S(\mathbb{R})$ with the codimension 1 subspace in the space of faster than polynomially decaying functions on $\mathbb{Z}^2$.

Can we "understand" (at least qualitatively) these two facts simultaneously?

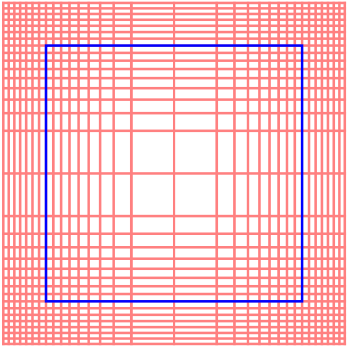
Geometric idea: distributions of type e^{ax^2+bx+c} (up to rescaling) where $a, b, c \in \mathbb{C}, \mathrm{Re}(a) < 0$ are in 1-to-1 correspondence with ellipses in $\mathbb{R}^2 = T^*\mathbb{R}$ of area 1. Limiting distributions as e.g. $\delta(x - x_0)$ or $e^{2\pi\mathrm{i}\,\xi_0 x}$ correspond to lines in $\mathbb{R}^2$.

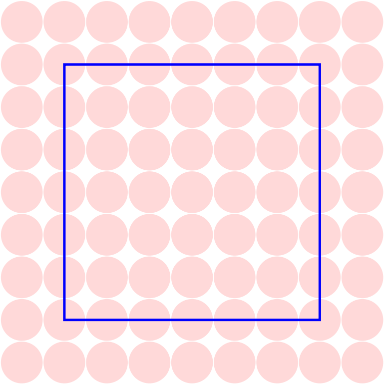


Hypothetical picture: pairings with complex gaussian functions or their limits determine function in $S(\mathbb{R})$ *uniquely* if for sufficiently large domains $\mathcal{U} \subset \mathbb{R}^2$ such that

$$\text{Length of } \partial\mathcal{U} \ll \text{Area of } \mathcal{U}$$

the number of corresponding ellipses intersecting $\mathcal{U}$ is at least $(1 + o(1)) \cdot \text{Area of } \mathcal{U}$.





Functions/sheaves/$\mathcal{D}$-modules

In many sitiations in mathematics, for an object $\mathcal{E}$ on a manifold X, its "true" support is not the usual closed subset $\mathrm{supp}(\mathcal{E}) \subseteq X$, but rather the **microlocal support** $\mu\mathrm{supp}(\mathcal{E}) \subseteq T^*X$.

Cotangent bundle T^*X carries a natural *symplectic* (i.e.non-degenerate and closed) 2-form ω.

The usual support for a non-trivial object could be very small, e.g. a point. On the contrary, microlocal support is always at least N-dimensional (if non-empty), where $N = \dim X$ (notice that $\dim T^*X = 2N$). Moreover, $\mu\mathrm{supp}\ \mathcal{E}$ is ***coisotropic***, i.e. the tangent space to it at any smooth point contains a Lagrangian subspace.

Microlocal support is conical along fibers of the projection $T^*X \to X$. In a situation with a small parameter it is no longer conical.

Examples

- (real analysis, L.Hörmander's theory of integral Fourier operators): for a real C^∞-manifold X, a large class of oscillating functions or distributions on X "correspond" to Lagrangian submanifolds of T^*X, e.g. $\exp(\frac{\mathrm{i}}{\hbar} \cdot f(x))$ where f is a real C^∞ function on X, and $\hbar \to +0$ is "Planck constant", corresponds to $L = \text{graph } df \subset T^*X$,
- often functions/distributions are solutions of a *holonomic* system of linear differential equations with polynomial coefficients, then one get a $\mathcal{D}_{X_\mathbb{C}}$-module $\mathcal{E}$ where $X_\mathbb{C}$ is a complex algebraic variety, and $\mu\text{supp } \mathcal{E}$ is a complex algebraic Lagrangian subvariety (possibly singular),
- by Riemann-Hilbert correspondence, in the case of regular singularities, one get a perverse constructible sheaf $\mathcal{F} = RH(\mathcal{E})$ on $X_\mathbb{C}$, with the same micro-support as of $\mathcal{E}$,
- one can consider (motivic) ℓ-adic constructible sheaves on algebraic varieties $X/\mathbb{F}_q$, corresponding $\overline{\mathbb{Q}}$-valued functions on $X(\mathbb{F}_{q^r})$ for $\forall r \geqslant 1$ are given by taking the trace of Frobenius.

Dimension of support, holonomic modules

Any finitely-generated *commutative* algebra $A \neq 0$ over a field $\mathbf{k}$ is of a *polynomial growth*, the dimension of $\operatorname{Spec} A$ ($=$ Krull dimension of A) can be calculated (for any given choice of finite number of generators) via

$$N = \dim \operatorname{Spec} A \iff \dim_{\mathbf{k}} A_{\leqslant n} \overset{n \to +\infty}{=} C \cdot n^N + O(n^{N-1}), \quad C > 0$$

where subspace $A_{\leqslant n} \subset A$ is spanned by products of $\leqslant n$ generators.
For a finitely-generated A-module $\mathcal{E} \neq 0$ one can calculate the dimension of support of $\mathcal{E}$ in a similiar way.

Any smooth affine variety $X/\mathbf{k}$ of dimension $N \geqslant 0 \rightsquigarrow$ associative algebra $D(X)/\mathbf{k}$ of differential operators. It is finitely-generated, and one has $\dim_{\mathbf{k}} D(X)_{\leqslant n} \overset{n \to +\infty}{=} C \cdot n^{2N} + O(n^{2N-1}), \quad C > 0.$
Fact: if $char(\mathbf{k}) = 0$, for any finitely-generated $D(X)$-module $\mathcal{E}$ its "dimension of support" $d(\mathcal{E})$ satisfies $\boxed{N \leqslant d(\mathcal{E}) \leqslant 2N}$.

$D(X)$-modules whose "dimension of support" is exactly equal to $N = \dim X$ are called ***holonomic***.

Usual notions of support

The definition of a holonomic $D(X)$-module which I gave, is *not the standard one*, except the case $X = \mathbb{A}^N_{\mathbf{k}}$. Here the algebra of differential operators (called *N-th Weyl algebra* $A_N = A_{N,\mathbf{k}}$) has canonical generators $x_1, \ldots x_N, \partial_1, \ldots, \partial_N$ subject to relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij} \,.$$

The associated filtration on A_N is called *Bernstein filtration*, its n-th term is spanned by $\vec{x}^{\vec{\alpha}} \vec{\partial}^{\vec{\beta}}$ where $|\vec{\alpha}| + |\vec{\beta}| \leqslant n$. For any finitely-generated module $\mathcal{E}$, after the choice of generators, we obtain a filtration by finite-dimensional subspaces. The associated graded module is a graded module over $\mathbf{k}[x_1, \ldots, x_N, y_1, \ldots, y_N]$ and its usual support (in the sense of algebraic geometry) is a ***conical*** coisotropic subvariety $\subseteq \mathbb{A}^{2N}_{\mathbf{k}}$.

For general smooth affine $X/\mathbf{k}$ it is more convenient to use the intrinsic filtration on $D(X)$ by the degree of differential operators. The support defined in this way is a closed coisotropic subset of T^*X, conical with respect to ***dilations*** along the fibers of projection $T^*X \to X$.

Category of holonomic modules as a filtered colimit

One can define supports in T^*X and holonomic D-modules for a general (non-affine) smooth variety $X/\mathbf{k}$. The subcategory Hol_X of holonomic $D(X)$-modules has very nice finiteness properties, and one can define the *Euler form* $\chi : K_0(Hol_X) \otimes K_0(Hol_X) \to \mathbb{Z}$ by

$$\chi([\mathcal{E}_1],[\mathcal{E}_2]) := \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbf{k}} Ext^i(\mathcal{E}_1, \mathcal{E}_2)\,.$$

Pathology: this form is of *infinite* rank (uncountable if $\#\mathbf{k} > \aleph_0$).

Claim: Category Hol_X is the colimit of full subcategories $Hol_{X,\leqslant\Sigma} \subset Hol_X$, where Σ belongs to certain filtered poset of "universal supports". For each subcategory $Hol_{X,\leqslant\Sigma}$ the corresponding Euler form has finite rank, and one can write a "topological" formula for it.

Morever, each $Hol_{X,\leqslant\Sigma}$ can be described as the heart of t-structure of a very nice class of triangulated categories: *the right orthogonal to an object in a saturated category*. For $\mathbf{k} = \mathbb{C}$ the same is true for the Betti side.

What kind of mathematical structure is Σ? In general, it will be a subset of certain set $\Sigma_\infty(T^*X, \omega_{st})$ defined as follows:

Definition

For an algebraic symplectic manifold (M, ω), define $\Sigma_\infty(M, \omega)$ to be the set of discrete valuations on the field of rational functions $\mathbf{k}(M)$ which are associated with a partial compactification $M' \supset M$ where $D := M' - M$ is a smooth irreducible divisor in M', such that in local formal coordinates divisor D is given by equation $x_1 = 0$ and the form ω is given by

$$\omega = \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^{\dim M/2} dx_i \wedge dy_i \,.$$

In the case $\dim X = 1$ the set $\Sigma_\infty(T^*X, \omega_{st})$ is the set of possible (ir)regular terms for meromorphic connections, i.e. formal expressions of the type

$$\exp\Big(\sum_{\lambda \in \mathbb{Q}_{<0}} c_\lambda (x - x_0)^\lambda\Big)$$

except regular term, $x_0 \notin X$; and $\Sigma \subset \Sigma_\infty(T^*X, \omega_{st})$ is *any* finite subset.

In general (for symplectic manifold (M,ω) of arbitrary dimension) one should proceed as follows. Consider a partial compactification $M' \supset M$ such that $M' - M = \cup_{i=1}^{k} D_i$ (here $1 \leqslant k \leqslant N$) is a simple normal crossing divisor such all intersections $\cap_{i\in I} D_i$ are non-empty and connected, and such that near a point of $D_1 \cap \cdots \cap D_k$ divisors D_i are given in some local formal coordinates by equations $x_i = 0$ and

$$\omega = \sum_{i=1}^{k} \frac{dx_i}{x_i} \wedge dy_i + \sum_{i=k+1}^{\dim M/2} dx_i \wedge dy_i \,.$$

Such a partial compactification $M' \supset M$ gives a ***rational simplex*** $\Delta_{\mathbb{Q}}^{k-1}(M') \subset \Sigma_\infty(M,\omega)$ consisting of exceptional divisors for all iterated blow-ups at intersections of $(D_i)_{1\leqslant i\leqslant k}$.

The sets $\Sigma \subset \Sigma_\infty(M,\omega)$ (parametrizing "good" full subcategories $Hol_{X,\leqslant\Sigma} \subset Hol_X$) are arbitrary finite unions of rational simplices. Each such a set is naturally identified with the set of $\mathbb{Q}$-points of a compact $\mathbb{Q}PL$-space of dimension $\leqslant \dim M/2 - 1$.

Quantum tori

There is a parallel story for the case of *quantum torus*, the algebra generated by $2N$ invertible elements

$$X_1^{\pm 1}, \dots X_N^{\pm 1}, Y_1^{\pm 1}, \dots, Y_N^{\pm 1}$$

subject to relations

$$X_i \cdot X_j = X_j \cdot X_i, \quad Y_i \cdot Y_j = Y_j \cdot Y_i, \quad Y_i \cdot X_j = q^{\delta_{ij}} \cdot X_j \cdot Y_i$$

where $q \in \mathbf{k}^\times$ is ***not a root of 1*** (e.g. it is a transcendental element of $\mathbf{k}$).

The standard strategy for defining supports of holonomic modules does not work. The new theory gives a very simple picture: start with $M := \mathbb{G}_m^{2N}$ with the symplectic form $\omega = \sum_{i=1}^N d\log x_i \wedge d\log y_i$; the set $\Sigma_\infty(M, \omega)$ coincides with the set of $\mathbb{Q}$-rays in $\mathbb{Q}^{2N}$. Supports of holonomic modules are sets of $\mathbb{Q}$-rays in rational polyhedral Lagrangian cones in $\mathbb{Q}^{2N}$.

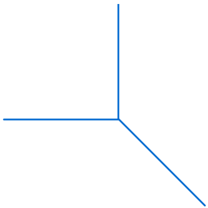
Modules over the algebra of functions on quantum torus encode q-difference equations. For example, the following infnite product (well-defined is $|q| < 1$)

$$f(x) = (1 - x)(1 - qx)(1 - q^2 x) \cdots = \sum_{n \geqslant 0} \frac{(-1)^n q^{n(n-1)/2}}{(1 - q) \ldots (1 - q^n)} x^n$$

solves the equation

$$f(x) = (1 - x) f(qx) \iff \big(1 - (1 - X)Y\big) f = 0$$

and the corresponding holonomic module has support



Local Riemann-Hilbert correspondence over $\mathbb{C}[[\hbar]]$

For a complex analytic symplectic manifold (M, ω) one has a *canonical* sheaf of abelian categories on M, which is locally is equivalent to the sheaf of categories of finitely-generated modules over algebras of type $\mathcal{O}(U)[[\hbar]]$ where $U \subset \mathbb{C}^{2N}_{x_1,\ldots,x_N,y_1,\ldots,y_N}$ is a Stein domain, endowed with the product

$$f \star g = \sum_{n \geqslant 0} \frac{\hbar^n}{n!} \sum_{1 \leqslant i_1,\ldots,i_n \leqslant N} \frac{\partial^n f}{\partial_{y_1} \ldots \partial_{y_n}} \cdot \frac{\partial^n g}{\partial_{x_1} \ldots \partial_{x_n}} \qquad ("y_i = \hbar \frac{\partial}{\partial x_i}")$$

For a (possibly singular) closed analytic Lagrangian subset $L \subset M$ one can define an abelian $\mathbb{C}((\hbar))$-linear category $Hol_{M,L}$ as the quotient of the category of finitely-generated modules supported on L by the Serre subcategory consisting of $\hbar$-torsion modules.

M.Kashiwara and P.Schapira proposed a description of Riemann-Hilbert type of certain full subcategory of $Hol_{M,L}$ in the case when L is smooth, consisting of objects which are locally isomorphic to finite sums of copies of the unique non-trivial simple object supported on L.

Even in the case of smooth L the total category $Hol_{M,L}$ is very complicated, a bit similar to the category of all holonomic D-modules when there is no formal parameter $\hbar$.

Together with Y.Soibelman, we found a correct language for the description of an "enhanced support" for a general object of $Hol_{M,L}$. Here one considers the Poisson variety $M \times \mathbb{C}_\hbar$ of dimension $2N+1$ endowed with the Poisson structure

$$\gamma = \hbar \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

in local Darboux coordinates $(x_1, \dots, x_N, y_1, \dots, y_N)$ on X. Then making blow-ups with smooth centers in the hypersurface $\hbar = 0$ (and also passing to ramified covers $\hbar \rightsquigarrow \hbar^{1/k}$ for $k \geqslant 1$) one arrives to the set $\Sigma_{sympl}(M, \omega)$ of exceptional divisors on which the extension of γ is non-singular and the divisor itself is a symplectic leaf. This is an analog of $\Sigma_\infty(M, \omega)$, and the generalized support is a compact $\mathbb{Q}PL$-space of dimension $\leqslant N$. Also we formulated a local Riemann-Hilbert correspondence in the case $N = 1$.

The enhanced supports should be used in asymptotic analysis. If we start with expressions of the type

$$e^{\frac{f(x)}{\hbar}} \cdot \left(g_0(x) + \hbar g_1(x) + \dots\right)$$

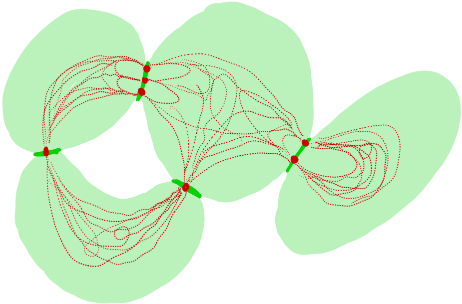
then after taking direct images (integrating over a part of variables), in certain non-generic situations we can arrive to expressions with the exponential part of the type

$$e^{\sum_{0<a/b<1} \hbar^{-a/b} f_{a/b}(x) + O(1)}$$

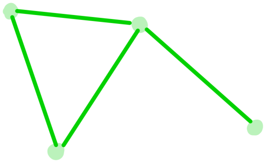
Roughly speaking, enhanced supports describe such sums of fractional powers of $\hbar$.

Picture of a partially anchored Lagrangian, $\dim M = 2$

symplectic leaves and lines of contact



support as a graph



Poisson structure ***vanishes*** on lines of contact $\Longrightarrow$ we have a well-defined intersection of "support" (an undetermined Lagrangian subvariety) with such divisors.

Towards a non-archimedean support

The supports of type $\Sigma \subset \Sigma_\infty(M,\omega)$ in the absolute case (no $\hbar$-dependence), or enhanced supports in the case of $\star$-products, are compact $\mathbb{Q}PL$ spaces. This reminds a notion of a *skeleton* (introduced long time ago by Y.Soibelman and myself in the context of mirror symmetry), for Calabi-Yau varieties (= varieties with a volume element) over a non-archimedean field. In general, the skeleton is closed subset of the Berkovich spectrum, consisting locally of multiplicative semi-norms on algebras of functions $\mathcal{O}(\mathcal{U})$ on non-archimedean Stein domains.

The set $\Sigma_{sympl}(M,\omega)$, mentioned in the case of $\star$-product, contains points corresponding to the usual (possibly singular) irreducible Lagrangian subvarieties $L \subset M$ (make the blow-up at $L \times \{0\} \subset M \times \mathbb{C}_\hbar$) as well as points of a different nature, e.g. in local coordinates the generic points of products of polydics

$$\{(\vec{x},\vec{y}) \in \overline{\mathbb{C}(\!(\hbar)\!)}^{2N} \,|\, |\vec{x}| \leqslant r_1, |\vec{y}| \leqslant r_2\}, \qquad r_1 r_2 = 1\,.$$

Such points remind ellipsoids from the beginning of my lecture, and give ***multiplicative*** semi-norms on $\star$-products. $1 \rightsquigarrow p^{\frac{-1}{p-1}}$ works also over $\mathbb{Q}_p$!

Proposal for Zariski spectrum based on p-supports

The category of holonomic modules is a bit deficient from the algebro-geometric point of view. A rude caricature is the category of coherent sheaves on $M = T^*X$ with a Lagrangian support. Let's be more ambitious, and try to find an analog of the full *Zariski spectrum* for the category of coherent (= finitely-generated) $D(X)$-modules. Again, a caricature will be the set of coisotropic points in the usual Zariski spectrum.

The trouble is that although algebra $D(X)$ has Ore property, and one can define the skew-field of fractions, one ***can not*** develop a theory of localizations parallelly to what we have in the commutative case. The basic pathological example is the 1-st Weyl algebra $\mathbf{k}[x][\partial_x]$ with inverted *both* elements x and ∂_x, i.e. with added two-sided inverses x^{-1}, ∂_x^{-1}. This algebra has *cubic* growth, instead of the expected quadratic one.

Here the positive characteristic comes to the rescue, and at least gives a clear picture of what is going wrong with naive localizations.

Recall that the microlocal supports of $D(X)$-modules have dimension $\geqslant \dim X$ in the $\mathrm{char}(\mathbf{k}) = 0$. This property fails totally in the case $\mathrm{char}(\mathbf{k}) = p > 0$, where the whole picture with Zariski spectrum easily makes sense. Indeed, consider for simplicity the N-th Weyl algebra

$$A_{N,R} := R[x_1, \dots, x_N][\partial_1, \dots, \partial_N]$$

when the coefficient ring R is over $\mathbb{Z}/p\mathbb{Z}$. Then this algebra is an Azumaya algebra of rank p^N (a twisted form of the matrix algebra) over its centrum

$$Z(A_{N,R}) = R[x_1^p, \dots, x_N^p, \partial_1^p, \dots, \partial_N^p] .$$

Let us now assume that R is a finitely-generated ring over $\mathbb{Z}$ without zero divisors, and $T \in A_{N,R}$ is a non-zero element. Then for all primes $p = 2, 3, 5, \dots$ element T gives an element $T \mod p$ of $A_{N,R/p}$, i.e. essentially a matrix-valied function on the $2N$-dimensional affine space. This matrix-valued function is invertible outside of the locus where its determinat vanishes, which is a hypersurface $Y_p \in \mathbb{A}^{2N}_{R/p}$, the *p-**support*** of the cyclic module $A_{N,R}/A_{N,R} \cdot T$.

The dependence of hypersurface Y_p on p is typically chaotic. First, its degree in grows like $O(p^{N-1})$ when $p \to +\infty$. And even in the case $N = 1$ when the degree stays constant, the resulting plane curve depends on p in an unpredictable manner. Hence we cannot control in general the intesections of p-supports!.

Nevertheless, there exist a class of ***special elements*** $T = \sum_{\vec{\alpha},\vec{\beta}} c_{\vec{\alpha},\vec{\beta}} \vec{x}^{\vec{\alpha}} \partial^{\vec{\beta}}$ such that there exists a *commutative* polynomial $\tilde{T} = \sum_{\vec{\alpha},\vec{\beta}} c_{\vec{\alpha},\vec{\beta}} \vec{x}^{\vec{\alpha}} y^{\vec{\beta}}$ with coefficients in $\mathbb{Q} \otimes R$, which can be called the ***arithmetic symbol*** of T, such that for all sufficiently large $p \gg 2$ one has

$$\det_{p^N \times p^N} T = \sum_{\vec{\alpha},\vec{\beta}} \tilde{c}^{p^N}_{\vec{\alpha},\vec{\beta}} (\vec{x}^p)^{p^{N-1}\vec{\alpha}} (\vec{\partial}^p)^{p^{N-1}\vec{\beta}} .$$

The class of special elements is quite large, it contains e.g. all polynomials in generators of degree $\leqslant 1$ (here the arithmetic symbol is the "same" affine-linear function), and some more complicated operators which commute to a large subalgebra of $A_{N,R}$ (typically some special linear combinations of commuting Hamiltoninas in a quantum integrable system).

If we consider complements to hypersurfaces given by $\tilde{T}(\vec{x}, \vec{y}) = 0$, it looks very reasonable to associate with them the localization of $A_{N,R}$ obtained by adding inverses of all special elements whose arythmetic symbol is a power of $\tilde{T}$.

A very speculative guess (it may well turn out to be wrong!) is that such new localized algebras (which are almost impossible to compute directly from the definition) behave similarly to what we have in the commutative world, one has descent property for covers etc.