Geometry and logic of subtoposes

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References:

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[Denseness] "Denseness conditions, morphisms and equivalences of toposes" by O. Caramello.

[TST] "Theories, Sites, Toposes" by O. Caramello.

[Engendrement] "Engendrement de topologies, démontrabilité et opérations sur les sous-topos" (to appear soon) by O. Caramello and L. Lafforgue.

I. Why subtoposes?

- **Why toposes?**
  - Toposes as a wide generalisation of topological spaces.
  - Toposes as universal invariants.
  - Toposes as pastiches of the category of sets.
  - Toposes as incarnations of the semantics of theories.
- **The multiple expressions of the notion of subtopos**
  - The categorical definition.
  - The expression in terms of Grothendieck topologies.
  - The logical expression in terms of quotient theories.
  - Provability as a topological problem.
- **The geometric operations on subtoposes**
  - Inner operations: intersection, union, difference.
  - Outer operations: existential push-forward,
    pull-back,
    universal push-forward.

Toposes as a wide generalisation of topological spaces:

Definition. – *A topos is a category $\mathcal{E}$ which is equivalent to the category*

$$\widehat{\mathcal{C}}_J \quad \text{of set-valued “sheaves”}$$

on a site $(\mathcal{C}, J)$ consisting in

$$\begin{cases} \mathcal{C} & = & \text{(essentially) small category,} \\ J & = & \text{“topology” on } \mathcal{C} = \text{notion of “covering”.} \end{cases}$$

Remark: Any topological space X defines the topos

$$\mathcal{E}_X = \text{category of set-valued sheaves on}$$

$$\begin{cases} \mathcal{C}_X & = & \text{category of open subsets of } X, \\ J_X & = & \text{ordinary notion of covering by subsets.} \end{cases}$$

Definition. – *A morphism of toposes $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ is a pair of adjoint functors*

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \, \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f^* respects finite limits.

Remarks:

- Any continous map $X' \xrightarrow{f} X$ induces a topos morphism $\qquad f : \mathcal{E}_{X'} \longrightarrow \mathcal{E}_X$.
- The map $(X' \xrightarrow{f} X) \mapsto (\mathcal{E}_{X'} \to \mathcal{E}_X)$ is one-to-one if X is “sober”.
- Points of a topos $\mathcal{E}$ are defined as topos morphisms $\qquad \{\text{sets}\} = \text{Pt} \longrightarrow \mathcal{E}$.

Toposes as universal invariants:

Cohomology:

Sheaf cohomology on topological spaces
generalises to arbitrary linear objets
of arbitrary toposes related by arbitrary morphisms of toposes.

Homotopy:

The construction of fundamental groups π_1
and higher homotopy groups π_i, $i \geq 2$,
of locally connected topological spaces X
factorises through their associated toposes $\mathcal{E}_X$
and generalises to toposes $\mathcal{E}$ which are "locally connected".

Topos invariants and Caramello's "bridge" theory:

- More generally, any construction or property which is

$$\begin{cases} - \quad \text{phrased in categorical terms,} \\ - \quad \text{well-defined for toposes (or wide classes of toposes),} \\ - \quad \text{invariant under equivalences of toposes,} \end{cases}$$

defines an invariant of sites $(\mathcal{C}, J)$.

- The expression of such an invariant in different equivalent sites

$$(\mathcal{C}, J) \text{ and } (\mathcal{C}', J') \quad \text{related by} \quad \widehat{\mathcal{C}}_J \cong \widehat{\mathcal{C}}'_{J'}$$

often generates unexpected equivalences.

Toposes as pastiches of the category of sets:

Theorem (Grothendieck-Giraud). – *A category $\mathcal{E}$ is a topos if and only if:*

(0) *$\mathcal{E}$ is locally small.*

(1) *Arbitrary limits are well-defined in $\mathcal{E}$.*

(2) *Arbitrary colimits are well-defined in $\mathcal{E}$.*

(3) *Base change functors $E' \times_E \bullet$ in $\mathcal{E}$ respect arbitrary colimits.*

(4) *Filtering colimit functors in $\mathcal{E}$ respect finite limits.*

(5) *Sums in $\mathcal{E}$ are disjoint.*

(6) *A morphism in $\mathcal{E}$ is an isomorphism if (and only if) it is a monomorphism and an epimorphism.*

(7) *Quotients of an objet E of $\mathcal{E}$ correspond one-to-one to equivalence relations $R \hookrightarrow E \times E$.*

(8) *The subobjects of an object E of $\mathcal{E}$ form a set.*

(9) *The quotients of an object E of $\mathcal{E}$ form a set.*

(10) *The contravariant functor $E \mapsto \{$subobjects of $E\}$ is representable by an object Ω of $\mathcal{E}$, the "subobject classifyer".*

(11) *For any objects E, E' of $\mathcal{E}$, the functor $\mathrm{Hom}(E \times \bullet, E')$ is representable by an object $\mathcal{H}om(E, E')$ of $\mathcal{E}$.*

(12) *The category $\mathcal{E}$ has small "separating" families of objects.*

Toposes for expressions of the semantics of theories:

Let $\mathbb{T}$ be a "geometric" first-order theory consisting in

- a vocabulary (or "signature")

$$\begin{cases} - & \text{names of objects (or "sorts"),} \\ - & \text{names of operations (or "function symbols"),} \\ - & \text{names of relations (or "relation symbols"),} \end{cases}$$

- a family of "axioms" taking the form of implications

$$\varphi(x_1^{A_1} \cdots x_n^{A_n}) \vdash \psi(x_1^{A_1} \cdots x_n^{A_n})$$

  between "formulas" in variables

$$x_1^{A_1} \cdots x_n^{A_n} \text{ associated with "sorts" } A_1, \cdots, A_n$$

  which are "geometric" in the sense that they only use the symbols

$$\begin{cases} \wedge \text{ (finite conjunction),} \quad \top \text{ (truth),} \\ \bigvee \text{ (arbitrary disjunction),} \quad \perp \text{ (false),} \\ \exists \text{ (existential quantifier in part of the variables).} \end{cases}$$

Proposition. – *(i) For any topos $\mathcal{E}$,*
there is a well-defined category of "models" of $\mathbb{T}$ in $\mathcal{E}$ $\quad \mathbb{T}\text{-mod}(\mathcal{E})$.
(ii) Any topos morphism $f : \mathcal{E}' \to \mathcal{E}$ induces a pull-back functor of models of $\mathbb{T}$

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}').$$

Toposes as incarnations of the semantics of theories:

Theorem (Grothendieck, Hakim, Lawvere, Joyal, Makkai, Reyes, $\cdots$). –
For any first-order geometric theory $\mathbb{T}$, there exist

$$\left\{ \begin{array}{l} \text{a topos } \mathcal{E}_{\mathbb{T}} \text{ (called the "classifying topos" of } \mathbb{T}\text{)}, \\ \text{a model } U_{\mathbb{T}} \text{ of } \mathbb{T} \text{ in } \mathcal{E}_{\mathbb{T}} \text{ (called the "universal model" of } \mathbb{T}\text{)} \end{array} \right.$$

such that, for any topos $\mathcal{E}$, the functor

$$\begin{array}{ccc} (\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* U_{\mathbb{T}} \\ \mathrm{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \\ \| & & \| \\ \left\{ \begin{array}{c} \text{category of} \\ \text{topos morphisms} \\ \mathcal{E} \to \mathcal{E}_{\mathbb{T}} \end{array} \right\} & & \left\{ \begin{array}{c} \text{category of} \\ \text{"models"} \\ \text{of } \mathbb{T} \text{ in } \mathcal{E} \end{array} \right\} \end{array}$$

is an equivalence of categories.

Remarks: • Conversely, for any topos $\mathcal{E}$,
there are infinitely many first-order geometric theories $\mathbb{T}$
such that $\mathcal{E}_{\mathbb{T}} \cong \mathcal{E}$.

• Theories $\mathbb{T}, \mathbb{T}'$ such that $\mathcal{E}_{\mathbb{T}} \cong \mathcal{E}_{\mathbb{T}'}$ can be called "semantically equivalent".

The multiple expressions of the notion of subtopos :

Categorical definition. –
A subtopos of a topos $\mathcal{E}$ is a full subcategory $\mathcal{E}' \hookrightarrow \mathcal{E}$ *such that:*

(1) *The embedding functor* $j_* : \mathcal{E}' \hookrightarrow \mathcal{E}$
has a left adjoint $j^* : \mathcal{E} \to \mathcal{E}'$.

(2) *This left adjoint* $j^* : \mathcal{E} \to \mathcal{E}'$
respects not only arbitrary colimits but also finite limits.

(3) *An object E of $\mathcal{E}$ belongs to the full subcategory $\mathcal{E}'$*
if and only if the canonical morphism

$$E \to j_* j^* E \qquad \text{is an isomorphism.}$$

Remarks:

• A topos morphism $f : \mathcal{E}' \to \mathcal{E}$
can be called an "embedding" if its push-forward component

$$f_* : \mathcal{E}' \to \mathcal{E} \qquad \text{is fully faithful}$$

or, equivalently, if the natural transformation

$$f^* \circ f_* \to \mathrm{Id}_{\mathcal{E}'} \qquad \text{is an isomorphism.}$$

• Subtoposes of a topos $\mathcal{E}$ can equivalently be defined
as equivalence classes of embeddings $\mathcal{E}' \hookrightarrow \mathcal{E}$.

Expressions of subtposes in terms of Grothendieck topologies:

Theorem (Grothendieck, SGA 4). –
Let $\mathcal{E}$ be a topos presented as the category of sheaves

$$\widehat{\mathcal{C}}_J \qquad \text{on a site } (\mathcal{C}, J)$$

consisting in an essentially small category $\mathcal{C}$ endowed with a topology J.
Then:

(i) *Any topology J' on $\mathcal{C}$ which contains J defines a subtopos*

$$\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J \cong \mathcal{E}\,.$$

(ii) *Conversely, any subtopos of $\widehat{\mathcal{C}}_J \cong \mathcal{E}$*
is associated with a unique topology $J' \supseteq J$ of $\mathcal{C}$.

Consequences:

- The subtoposes of any topos $\mathcal{E}$ form a partially ordered set.
- Arbitrary joins $\bigvee$ of subtoposes are always well-defined.
  They correspond to arbitrary intersections of topologies.
- Arbitrary intersections $\bigwedge$ of subtoposes are always well-defined.
  They correspond to topologies generated
  by families of topologies.

Logical expression of subtoposes in terms of quotient theories:

Definition. – *Let $\mathbb{T}$ be a geometric first-order theory written in a vocabulary Σ. Then:*

(i) *A quotient theory $\mathbb{T}'$ of $\mathbb{T}$ is a geometric first-order theory written in the same vocabulary Σ and such that any implication (or "sequent") of geometric formulas*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$$

which is provable in $\mathbb{T}$ is also provable in $\mathbb{T}'$.

(ii) *Two quotient theories $\mathbb{T}_1$ and $\mathbb{T}_2$ of $\mathbb{T}$ are called "syntactically equivalent" if they have the same provable implications* $\varphi(\vec{x}) \vdash \psi(\vec{x})$.

Theorem (O.C., PhD thesis; see chapter 3 of [TST]). –

(i) *Any quotient theory $\mathbb{T}'$ of a geometric first-order theory $\mathbb{T}$ is classified by a subtopos* $\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}}$.

(ii) *Conversely, any subtopos* $\mathcal{E}' \hookrightarrow \mathcal{E}_{\mathbb{T}}$ *is associated with a quotient theory $\mathbb{T}'$ of $\mathbb{T}$, which is unique up to syntactic equivalence.*

Provability as a topological problem:

Corollary. –

Suppose $\mathbb{T}$ *is a geometric first-order theory written in a vocabulary* Σ
and its classifying topos $\mathcal{E}_{\mathbb{T}}$
is presented as the category of sheaves on a site $(\mathcal{C}, J)$*:* $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J$.
Then it is possible to construct from any sequent of geometric formulas

$\varphi(\vec{x}) \vdash \psi(\vec{x})$ *in the vocabulary* Σ

a family of "sieves" on $\mathcal{C}$ $\mathcal{X}_{\vec{x},\varphi,\psi}$
such that:

(i) *For any quotient theory* $\mathbb{T}'$ *of* $\mathbb{T}$
defined by a family of extra axioms $\varphi_i(\vec{x}_i) \vdash \psi_i(\vec{x}_i)$, $i \in I$,
the associated subtopos $\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J$
corresponds to the topology $J' \supseteq J$ *on* $\mathcal{C}$
generated by J *and the families of sieves* $\mathcal{X}_{\vec{x}_i,\varphi_i,\psi_i}$, $i \in I$.

(ii) *Any implication of geometric formulas* $\varphi(\vec{x}) \vdash \psi(\vec{x})$
is provable in $\mathbb{T}'$ *if and only if*
all sieves in the associated family $\mathcal{X}_{\vec{x},\varphi,\psi}$
belong to the topology J' *generated by* J *and the families* $\mathcal{X}_{\vec{x}_i,\varphi_i,\psi_i}$.

First inner geometric operations on subtoposes:

Lemma. – *Let $\mathcal{E}$ be a topos and $(\mathcal{E}_i \hookrightarrow \mathcal{E})_{i \in I}$ a family of subtoposes.*

(i) *There exists a unique subtopos* $\bigvee_{i \in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ *characterized by the property that, for any subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$,*

$$\mathcal{E}' \supseteq \bigvee_{i \in I} \mathcal{E}_i \Longleftrightarrow \mathcal{E}' \supseteq \mathcal{E}_i, \quad \forall\, i \in I\,.$$

(ii) *There exists a unique subtopos* $\bigwedge_{i \in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ *characterized by the property that, for any subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$,*

$$\mathcal{E}' \subseteq \bigwedge_{i \in I} \mathcal{E}_i \Longleftrightarrow \mathcal{E}' \subseteq \mathcal{E}_i, \quad \forall\, i \in I\,.$$

Remark: If $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ and the subtoposes $\mathcal{E}_i \hookrightarrow \mathcal{E}$ are associated with topologies $J_i \supseteq J$, $i \in I$, then:

- the subtopos $\bigvee_{i \in I} \mathcal{E}_i$ is associated with the topology $$\bigcap_{i \in I} J_i\,,$$
- the subtopos $\bigwedge_{i \in I} \mathcal{E}_i$ is associated with the topology generated by the J_i's, $i \in I$.

The inner operation of difference of subtoposes:

Proposition (Joyal?; see [Elephant]). –
For any pair of subtoposes $\mathcal{E}_1, \mathcal{E}_2$ of a topos $\mathcal{E}$,
there exists a unique subtopos
$$\mathcal{E}_1 \backslash \mathcal{E}_2 \hookrightarrow \mathcal{E}$$
characterized by the property that, for any subtopos $\mathcal{E}' \hookrightarrow \mathcal{E}$,
$$\mathcal{E}_1 \backslash \mathcal{E}_2 \subseteq \mathcal{E}' \Longleftrightarrow \mathcal{E}_1 \subseteq \mathcal{E}_2 \vee \mathcal{E}' .$$

Remark:
If $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ and $\mathcal{E}_1, \mathcal{E}_2$ are defined by topologies J_1, J_2 on $\mathcal{C}$,
then $\mathcal{E}_1 \backslash \mathcal{E}_2$ is defined by a topology denoted
$$(J_2 \Rightarrow J_1)$$
and characterized by the property that, for any topology J' on $\mathcal{C}$,
$$(J_2 \Rightarrow J_1) \supseteq J' \Longleftrightarrow J_1 \supseteq (J_2 \cap J').$$

Corollary. – *For any subtopos $\mathcal{E}' \hookrightarrow \mathcal{E}$ of a topos $\mathcal{E}$, we have:*

(i) *The map of union with $\mathcal{E}'$* $\qquad \mathcal{E}' \vee \bullet$
respects arbitrary intersections of subtoposes of $\mathcal{E}$.

(ii) *The map of intersection with $\mathcal{E}'$* $\qquad \mathcal{E}' \wedge \bullet$
respects finite unions of subtoposes of $\mathcal{E}$.

Images of topos morphisms:

Proposition. – *Any topos morphism* $f : \mathcal{E}' \to \mathcal{E}$ *uniquely factorizes as*

$$\mathcal{E}' \xrightarrow{\bar{f}} \mathrm{Im}(f) \xhookrightarrow{j_f} \mathcal{E}$$

where

$$\begin{cases} \bullet \quad \mathrm{Im}(f) \xhookrightarrow{j_f} \mathcal{E} \text{ is an embedding of a subtopos,} \\ \bullet \quad \mathcal{E}' \xrightarrow{\bar{f}} \mathrm{Im}(f) \text{ is a "surjective" topos morphism} \\ \quad \text{in the sense that its pull-back component} \\ \qquad \bar{f}^* : \mathrm{Im}(f) \longrightarrow \mathcal{E}' \quad \text{is faithful.} \end{cases}$$

Remarks:

- If $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ and $\mathcal{C}$ is endowed with $\mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$,
  the subtopos $\mathrm{Im}(f) \hookrightarrow \mathcal{E}$ is defined by the topology $J' \supseteq J$
  for which a sieve $S \hookrightarrow y(X)$ is covering if and only if its transform by
  $\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J \cong \mathcal{E} \xrightarrow{f^*} \mathcal{E}'$ is an isomorphism of $\mathcal{E}'$.
- If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ and $f : \mathcal{E}' \to \mathcal{E}$ corresponds to a model M of $\mathbb{T}$ in $\mathcal{E}'$,
  the subtopos $\mathrm{Im}(f) \hookrightarrow \mathcal{E}$ corresponds to the "theory of M" $\mathbb{T}_M$
  i.e. the quotient theory $\mathbb{T}_M$ of $\mathbb{T}$ for which a sequent $\varphi(\vec{x}) \vdash \psi(\vec{x})$
  is provable in $\mathbb{T}_M$ if and only if it is verified by M.

Existential push-forward and pull-back of subtoposes:

Proposition. – *Let $f : \mathcal{E}' \to \mathcal{E}$ be a morphism of toposes.*
Then:

(i) *The map*

$$f_* : \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\} \longrightarrow \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\},$$

$$(\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto (\mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \xrightarrow{f} \mathcal{E}) \hookrightarrow \mathcal{E})$$

respects the order relation $\supseteq$
and arbitrary unions of subtoposes.

(ii) *Equivalently, it has a left adjoint*

$$f^{-1} : \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\} \longrightarrow \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\},$$

$$(\mathcal{E}_1 \hookrightarrow \mathcal{E}) \longmapsto (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}')$$

characterized by the property that,
for any subtoposes $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ and $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$,

$$f^{-1}\mathcal{E}_1 \supseteq \mathcal{E}'_1 \Leftrightarrow \mathcal{E}_1 \supseteq f_*(\mathcal{E}'_1) = \mathrm{Im}(\mathcal{E}'_1)\,.$$

Remark:
The map f^{-1}
respects the order relation $\supseteq$
and arbitrary intersections of subtoposes.

Universal push-forward of subtoposes:

Theorem (O.C., L.L., to appear in [Engendrement]). –
Let $f : \mathcal{E}' \to \mathcal{E}$ be a topos morphism which is "locally connected". Then:

(i) *The associated pull-back map*

$$f^{-1} : \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\} \longrightarrow \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\},$$
$$(\mathcal{E}_1 \hookrightarrow \mathcal{E}) \longmapsto (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}')$$

respects arbitrary unions of subtoposes.

(ii) *Equivalently, it has a left adjoint*

$$f_! : \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\} \longrightarrow \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\},$$
$$(\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto (f_!\mathcal{E}'_1 \hookrightarrow \mathcal{E}')$$

characterized by the property that,
for any subtoposes $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ and $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$,

$$f_!\mathcal{E}'_1 \supseteq \mathcal{E}_1 \Leftrightarrow \mathcal{E}'_1 \supseteq f^{-1}\mathcal{E}_1 \,.$$

Corollary. – *For any topos morphism $f : \mathcal{E}' \to \mathcal{E}$,*
the associated pull-back map

$$f^{-1} : \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\} \longrightarrow \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\}$$

respects finite unions of subtoposes.

II. Subtoposes and Grothendieck topologies:

- **The general notion of Galois connection**
  - Equivalences induced by pairs of adjoint functors.
  - The particular case of ordered structures.
  - Pairs of adjoint functors defined by relations.
  - Induced equivalences and generation processes.
- **The duality of sieves and presheaves**
  - Definition of their relation.
  - The induced duality of topologies and subtoposes.
  - Grothendieck topologies as fixed points.
  - Subtoposes as fixed points.
- **The duality of monomorphisms and objects in a topos**
  - Definition of their relation.
  - The induced notion of topology on a topos.
  - The induced duality of topologies and subtoposes.
- **The duality of sieves and monomorphisms of presheaves**
  - Definition of their relation.
  - The induced duality fo topologies and closedness properties.
  - Topologies as fixed points.
  - Closedness properties as fixed points.

Equivalences induced by pairs of adjoint functors:

Proposition. – *Consider a pair of adjoint functors*

$$(\mathcal{C} \xrightarrow{F} \mathcal{D}\,,\ \mathcal{D} \xrightarrow{G} \mathcal{C})$$

between locally small categories.
Let $\mathcal{C}'$ [resp. $\mathcal{D}'$] be the full subcategory of $\mathcal{C}$ [resp. $\mathcal{D}$] on "fixed points", i.e. objects X of $\mathcal{C}$ [resp. Y of $\mathcal{D}$] such that the canonical adjunction morphism

$$X \longrightarrow G \circ F(X) \qquad \text{[resp. } F \circ G(Y) \to Y\text{]}$$

is an isomorphism.
Then F and G induce converse equivalences

$$\mathcal{C}' \underset{\sim}{\overset{\sim}{\rightleftarrows}} \mathcal{D}'\,.$$

Proof: If $X \to G \circ F(X)$ is an isomorphism and $Y = F(X)$, then $Y \to F \circ G(Y)$ is also an isomorphism.
This implies that the canonical morphism

$$F \circ G(Y) \to Y \quad \text{is an isomorphism}$$

as the composite

$$F(X) \to F \circ G \circ F(X) \to F(X) \quad \text{is} \quad \mathrm{Id}_{F(X)}.$$

The particular case of ordered structures:

Corollary. – *Consider a pair of partially ordered sets or classes related by a pair of order-preserving maps*

$$(C, \leq) \underset{G}{\overset{F}{\rightleftarrows}} (D, \leq)$$

which are adjoint in the sense that

$$F(c) \leq d \Leftrightarrow c \leq G(d), \quad \forall\, c \in C,\ \forall\, d \in D.$$

Then:

(i) *If $C' = \{c \in C \mid G \circ F(c) = c\}$ and $D' = \{d \in D, F \circ G(d) = d\}$,*

F and G induce inverse bijections $C' \underset{\sim}{\overset{\sim}{\rightleftarrows}} D'$.

(ii) *An element $c \in C$ [resp. $d \in D$] is fixed by $G \circ F$ [resp. by $F \circ G$] if and only if it is an image in the sense that*

$$c \in \mathrm{Im}(G) \qquad \text{[resp. } d \in \mathrm{Im}(F)\text{].}$$

Remarks: For any $c \in C$ [resp. $d \in D$], we have

$$c \leq G \circ F(c) \qquad \text{[resp. } F \circ G(d) \leq d\text{]}$$

and $G \circ F(c) \leq c'$ if $c' \in C'$ and $c \leq c'$
[resp. $d' \leq F \circ G(d)$ if $d' \in D'$ and $d' \leq d$].
Proof of (ii): If $c = G(d)$, we have

$$c \leq (G \circ F)(c) = G \circ (F \circ G)(d) \leq G(d) = c.$$

Pairs of adjoint maps defined by relations:

Lemma (O.C., extracted from [Engendrement]). –
Consider an arbitrary relation $R \hookrightarrow T \times S$
between a pair of sets or classes T *and* S.
Then R *defines a pair of adjoint order-preserving maps*

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq)$$

between the partially ordered sets or classes of subsets or subclasses of S *and* T

$$F_R(J) = \{s \in S \mid (t,s) \in R,\ \forall\, t \in J\} \text{ for any } J \subseteq T,$$
$$G_R(I) = \{t \in T \mid (t,s) \in R,\ \forall\, s \in I\} \text{ for any } I \subseteq S.$$

Proof:

- It is obvious on the definition that

$$J_1 \subseteq J_2 \quad \Rightarrow \quad F_R(J_1) \supseteq F_R(J_2),$$
$$I_1 \supseteq I_2 \quad \Rightarrow \quad G_R(I_1) \subseteq G_R(I_2).$$

- For $J \subseteq T$ and $I \subseteq S$, we have equivalences

$$\begin{aligned} F_R(J) \supseteq I \quad &\Leftrightarrow \quad (t,s) \in R,\ \forall\, t \in J,\ \forall\, s \in I \\ &\Leftrightarrow \quad J \subseteq G_R(I). \end{aligned}$$

Induced equivalences and generation processes:

Corollary. – *Consider a relation* $R \hookrightarrow T \times S$ *and the induced pair of adjoint order-preserving maps*

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq).$$

Then:

(i) *The maps* F_R *and* G_R *induce inverse bijections*

$$\{J \subseteq T \mid G_R \circ F_R(J) = J\} \rightleftarrows \{I \subseteq S \mid F_R \circ G_R(I) = I\}.$$

(ii) *For any* $J \subseteq T$ *[resp.* $I \subseteq S$*], we have*

$$G_R \circ F_R(J) = J \qquad \text{[resp. } F_R \circ G_R(I) = I\text{]}$$

if and only if there exists $I \subseteq S$ *[resp.* $J \subseteq T$*] such that*

$$J = G_R(I) \qquad \text{[resp. } I = F_R(J)\text{]}.$$

(iii) *For any* $J \subseteq T$ *[resp.* $I \subseteq S$*], we have*

$$J \subseteq G_R \circ F_R(J) \qquad \text{[resp. } F_R \circ G_R(I) \supseteq I\text{]}$$

and $\quad J \subseteq J' \Rightarrow G_R \circ F_R(J) \subseteq J'$ *if* $J' = G_R \circ F_R(J')$

[resp. $I' \supseteq I \Rightarrow I' \supseteq F_R \circ G_R(I)$ *if* $I' = F_R \circ G_R(I')$*].*

Remark: For any $J \subseteq T$ [resp. $I \subseteq S$], $G_R \circ F_R(J)$ [resp. $F_R \circ G_R(I)$] can be called the element of $\mathrm{Im}(G_R)$ [resp. of $\mathrm{Im}(F_R)$] generated by J [resp. I].

The duality of sieves and presheaves:

Definition. –
Consider an essentially small category $\mathcal{C}$, *endowed with the Yoneda functor*

$$y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}} = \{\textit{category of presheaves} \quad \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}\}.$$

Let T *be the class of pairs* (X, C) *consisting in*

$$\begin{cases} X = \textit{object of } \mathcal{C}, \\ C = \textit{sieve on } X = \textit{subpresheaf of } y(X). \end{cases}$$

Let S *be the class of presheaves* P *on* $\mathcal{C}$.
We shall call "duality of sieves and presheaves" the relation

$$R \hookrightarrow T \times S$$

consisting in pairs of elements

$$(C \hookrightarrow y(X), P)$$

such that, for any morphism $X' \xrightarrow{x} X$ *of* $\mathcal{C}$, *the restriction map*

$$P(X') = \mathrm{Hom}(y(X'), P) \longrightarrow \mathrm{Hom}(C \times_{y(X)} y(X'), P)$$

is one-to-one.

Consequence: This relation induces adjoint order-preserving maps

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq).$$

The induced duality of topologies and subtoposes:

Theorem (extracted from [Engendrement]). –

(i) *A subclass J of $T = \{$sieves C on objects X of $\mathcal{C}\}$ is a fixed point of the duality of T with $S = \{$presheaves P on $\mathcal{C}\}$ if and only if J is a Grothendieck topology.*

(ii) *A subclass I of S is a fixed point of the duality of T and S if and only if I is the class of objects of a subtopos $\mathcal{E}$ of $\widehat{\mathcal{C}}$.*

Corollary. –

(i) *The duality of sieves and presheaves on $\mathcal{C}$ induces a one-to-one correspondence between Grothendieck topologies J on $\mathcal{C}$ and subtoposes of $\widehat{\mathcal{C}}$.*

(ii) *For any family J of sieves, $G_R \circ F_R(J)$ is the topology generated by J.*

(iii) *For any class I of presheaves on $\mathcal{C}$, $F_R \circ G_R(I)$ is the smallest subtopos of $\widehat{\mathcal{C}}$ which contains I.*

Precise identification of fixed points:

Theorem. – *A class $J \subseteq T$ of sieves $(C \hookrightarrow y(X))$ on $\mathcal{C}$ is a fixed point of the duality of T with S if and only if it is a topology, i.e. verifies:*

(Max) For any object X of $\mathcal{C}$, the maximal sieve $y(X)$ belongs to J.

(Stab) If $(C \hookrightarrow y(X))$ belongs to J, then for any morphism $x : X' \to X$,
$$(x^* C = C \times_{y(X)} y(X') \hookrightarrow y(X')) \text{ also belongs to } J.$$

(Trans) If $(C' \hookrightarrow y(X))$ belongs to J, a sieve $(C \hookrightarrow y(X))$ belongs to J if, for any morphism $X' \xrightarrow{x} X$ belonging to C',
$$(x^* C = C \times_{y(X),x} y(X') \hookrightarrow y(X')) \text{ belongs to } J.$$

Theorem. – *A class $I \subseteq S$ of presheaves on $\mathcal{C}$ is a fixed point of the duality if and only if the full subcategory $\mathcal{E}$ of $\widehat{\mathcal{C}}$ on objects of I is a "subtopos" in the sense that:*

(1) *The embedding functor $\mathcal{E} \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}$ has a left-adjoint j^*.*

(2) *This left-adjoint $j^* : \widehat{\mathcal{C}} \to \mathcal{E}$ respects finite limits.*

(3) *An object P of $\widehat{\mathcal{C}}$ is in $\mathcal{E}$, i.e. is an element of I, if and only if the canonical morphism $P \longrightarrow j_* \circ j^* P$ is an isomorphism.*

Any class of presheaves defines a Grothendieck topology:

• Consider a class I of presheaves P on $\mathcal{C}$.
We need to verify that the class J of sieves C on objects X of $\mathcal{C}$ such that

$$\begin{cases} \text{for any morphism } X' \xrightarrow{x} X \text{ and any } P \in I, \\ \text{the restriction map } P(X') = \mathrm{Hom}(y(X'), P) \longrightarrow \mathrm{Hom}(C \times_{y(X)} y(X'), P) \text{ is one-to-one} \end{cases}$$

is a topology.

• Any intersection of topologies is a topology.
So it is enough to consider the case where I has a unique element P.

• The above condition is verified by maximal sieves $C = y(X)$.

• By definition, it is stable under base change by any $X' \xrightarrow{x} X$.

• Consider sieves $C \hookrightarrow y(X)$ and $C' \hookrightarrow y(X)$
such that $C' \in J$ and $x^*C = C \times_{y(X)} y(X') \in J$, $\forall\, (X' \xrightarrow{x} X) \in C'$.
As these conditions are respected by base change,
we are reduced to check that the map

$$P(X) = \mathrm{Hom}(y(X), P) \longrightarrow \mathrm{Hom}(C, P) \text{ is one-to-one.}$$

For any morphism $C \xrightarrow{p} P$, the composite induced by any $(X' \xrightarrow{x} X) \in C'$

$$x^*C = C \times_{y(X)} y(X') \longrightarrow C \longrightarrow P$$

uniquely lifts to a morphism $y(X') \to P$. This defines a morphism $C' \to P$
which lifts to $y(X) \to P$. The composite $C \hookrightarrow y(X) \to P$ coincides with $C \xrightarrow{p} P$
as they coincide on $C \times_{y(X)} C'$.

Any class of sieves defines a subtopos:

• If J is a class of sieves $C \hookrightarrow y(X)$ on objects X of $\mathcal{C}$,
the class of presheaves $F_R(J)$ associated with J
is the same as the class of presheaves associated with $G_R \circ F_R(J)$.
• So we may suppose that J is a topology.
• Then $F_R(J)$ is the class of J-sheaves on $\mathcal{C}$.
The full subcategory $\widehat{\mathcal{C}}_J$ on $F_R(J)$ is a subtopos

$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J, \widehat{\mathcal{C}}_J \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}).$$

• Furthermore, in that case, J is the class of sieves $C \hookrightarrow y(X)$ such that

$$j^* C \longrightarrow j^* \circ y(X) \quad \text{is an isomorphism}$$

or, equivalently, such that for any J-sheaf E,
the restriction map

$$\mathrm{Hom}(y(X), E) \xrightarrow{\sim} \mathrm{Hom}(C, E) \quad \text{is one-to-one}.$$

• This proves that, if J is a topology,

$$G_R \circ F_R(J) = J\,.$$

• In other words, a subclass J of $T = \{\text{sieves } C \hookrightarrow y(X)\}$
is a fixed point of $G_R \circ F_R$
if and only if it is a topology.

Any subtopos is a fixed point of the duality relation:

- Consider a subtopos of $\widehat{\mathcal{C}}$ defined by a class I of presheaves
$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \mathcal{E} \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}).$$
- For any sieve $C \hookrightarrow y(X)$, the restriction map
$$\mathrm{Hom}(y(X), j_*E) \longrightarrow \mathrm{Hom}(C, j_*E) \quad \text{is one-to-one}$$
for any object E of $\mathcal{E}$ if and only if the induced morphism
$$j^*C \longrightarrow j^*y(X) \text{ is an isomorphism.}$$
- This condition defines a topology $J = G_R(I)$.
- We have to check that, conversely, any J-sheaf E is an object of $\mathcal{E}$, i.e. verifies $E \xrightarrow{\sim} j_* \circ j^*E$.
- Consider the diagonal embedding $E \hookrightarrow E \times_{j_* \circ j^* E} E$.
For any morphism $y(X) \to E \times_{j_* \circ j^* E} E$,
its fiber product with the diagonal E is a sieve
$$C \hookrightarrow y(X) \text{ which belongs to } J.$$
So the morphism $C \to E$ uniquely lifts to $y(X) \to E$.
- This proves that $E \to j_* \circ j^*E$ is a monomorphism.
Its fiber product with any morphism $y(X) \to j_* \circ j^*E$
is a sieve $\quad C \longhookrightarrow y(X) \quad$ which belongs to J.
- So the morphism $C \to E$ uniquely lifts to $y(X) \to E$ which means that
$$E \longrightarrow j_* \circ j^*E \quad \text{is an isomorphism.}$$

The duality of monomorphisms and objects in a topos:

Definition. –
Consider a topos $\mathcal{E}$.
Consider the class T of monomorphisms of $\mathcal{E}$

$$C \hookrightarrow X.$$

Consider the class S of objects E of $\mathcal{E}$.
We shall call "duality of monomorphisms and objects" in $\mathcal{E}$ the relation

$$R \hookrightarrow T \times S$$

consisting in pairs of elements

$$(C \hookrightarrow X, E)$$

such that, for any morphism $X' \to X$ of $\mathcal{E}$,
the restriction map

$$\mathrm{Hom}(X', E) \longrightarrow \mathrm{Hom}(C \times_X X', E) \quad \text{is one-to-one.}$$

This relation induces a pair of adjoint order-preserving maps

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq).$$

The induced notion of topology on a topos:

Proposition. –
A subclass $J \subseteq T = \{$*monomorphisms* $C \hookrightarrow X$ *of* $\mathcal{E}\}$
is a fixed point of the duality of T *and* $S = \{$*objects of* $\mathcal{E}\}$
if and only if it is a topology of $\mathcal{E}$
in the sense that it verifies the conditions:

$$\begin{cases} \text{(Max)} & \text{Any isomorphism } C \xrightarrow{\sim} X \text{ is an element of } J. \\ \text{(Stab)} & \text{Base change by any morphism } X' \to X \text{ of } \mathcal{E} \\ & \text{transforms elements } C \hookrightarrow X \text{ of } J \\ & \text{into elements } C \times_X X' \hookrightarrow X' \text{ of } J. \\ \text{(Trans)} & \text{A monomorphism} \\ & \qquad\qquad C \hookrightarrow X \quad \text{is in } J \\ & \text{if there exist} \quad (C' \hookrightarrow X) \in J \\ & \text{and a globally epimorphic family} \quad (X_k \to C')_{k \in K} \\ & \text{such that all fiber products} \\ & \qquad C \times_X X_k \hookrightarrow X_k\,, \quad k \in K\,, \quad \text{belong to } J. \end{cases}$$

The induced notion of subtopos of a topos:

Proposition. –
A subclass $I \subseteq S = \{objects\ E\ of\ \mathcal{E}\}$
is a fixed point of the duality of S *and* $T = \{monomorphisms\ of\ \mathcal{E}\}$
if and only if the full subcategory $\mathcal{E}_I$ *of* $\mathcal{E}$ *on objects of* I
is a subtopos
in the sense that it verifies the conditions:

$$\begin{cases} (1) & \text{The embedding functor } j_* : \mathcal{E}_I \hookrightarrow \mathcal{E} \text{ has a left adjoint } j^*. \\ (2) & \text{This left adjoint functor } j^* : \mathcal{E} \to \mathcal{E}_I \\ & \text{respects finite limits.} \\ (3) & \text{An objet } E \text{ of } \mathcal{E} \text{ belongs to } I \\ & \text{if and only if the canonical morphism} \\ & \qquad E \longrightarrow j_* \circ j^* E \quad \text{is an isomorphism.} \end{cases}$$

The induced duality of topologies and subtoposes:

We still consider the pair of adjoint order-preserving maps

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq)$$

defined by the duality R of $T = \{\text{monomorphisms of } \mathcal{E}\}$
and $S = \{\text{objets of } \mathcal{E}\}$.

Corollary. –

(i) *This duality induces a one-to-one correspondence between topologies on the topos $\mathcal{E}$*

and subtoposes $(\mathcal{E} \xrightarrow{j^*} \mathcal{E}', \mathcal{E}' \overset{j_*}{\hookrightarrow} \mathcal{E})$ of $\mathcal{E}$.

(ii) *For any subclass J of monomorphisms $C \hookrightarrow X$ of $\mathcal{E}$,
$G_R \circ F_R(J)$ is the topology generated by J,
i.e. the smallest topology which contains J.*

(iii) *For any subclass I of objects E of $\mathcal{E}$,
$F_R \circ G_R(I)$ is the subtopos generated by I,
i.e. the smallest subtopos of $\mathcal{E}$ which contains I.*

The duality of sieves and monomorphisms of presheaves:

Definition. –
Consider an essentially small category $\mathcal{C}$, endowed with $y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$.
Consider the class $T = \{\text{sieves } C \hookrightarrow y(X)\}$.
Consider the class S of monomorphisms of presheaves on $\mathcal{C}$

$$Q \hookrightarrow P\,.$$

We shall call "duality of sieves and subpresheaves" on $\mathcal{C}$ the relation

$$R \hookrightarrow T \times S$$

consisting in pairs of elements

$$(C \hookrightarrow y(X),\ Q \hookrightarrow P)$$

such that:

$$\begin{cases} \text{for any morphism } X' \xrightarrow{x} X \text{ of } \mathcal{C} \text{ and any element } p \in P(X'), \\ \text{one has } p \in Q(X') \\ \text{if } x'^*(p) \in Q(X'')\,,\ \forall\,(X'' \xrightarrow{x'} X') \in x^* C. \end{cases}$$

This relation induces a pair of adjoint order-preserving maps

$$(\mathcal{P}(T), \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (\mathcal{P}(S), \supseteq).$$

Topologies and closedness properties as fixed points:

Theorem (extracted from [Engendrement]). –
(i) A subclass $J \subseteq T = \{$sieves $C \hookrightarrow y(X)\}$ is a fixed point of the duality of T and $S = \{$subpresheaves $Q \hookrightarrow P\}$ if and only if J is a topology on $\mathcal{C}$.
(ii) A subclass $I \subseteq S$ is a fixed point of the duality of T and S if and only if I is a "closedness property" in the sense that if verifies the following conditions:

$$
\begin{cases}
(1) & \text{Isomorphisms } Q \xrightarrow{\sim} P \text{ belong to } I. \\
(2) & \text{Base change by any morphism } P' \to P \text{ of } \widehat{\mathcal{C}} \\
 & \text{transforms subpresheaves } Q \hookrightarrow P \text{ which belong to } I \\
 & \text{into subpresheaves } Q \times_p P' \hookrightarrow P' \text{ which belong to } I. \\
(3) & \text{For any family of subpresheaves} \\
 & \qquad Q_k \longhookrightarrow P\,,\ k \in K\,, \text{ which belong to } I, \\
 & \text{their intersection} \\
 & \qquad \bigcap_{k \in K} Q_k \longhookrightarrow P \text{ still belongs to } I. \\
(4) & \text{If } \overline{Q} \hookrightarrow P \text{ denotes the smallest element of } I \text{ containing some } Q \hookrightarrow P, \\
 & \text{one has for any morphism } P' \to P \text{ of } \widehat{\mathcal{C}} \\
 & \qquad \overline{P' \times_P Q} = P' \times_P \overline{Q}\,.
\end{cases}
$$

The induced duality of topologies and closedness properties:

Corollary. –

(i) *The "duality of sieves and subpresheaves" on $\mathcal{C}$ induces a one-to-one correspondence between Grothendieck topologies of $\mathcal{C}$ and closedness properties on subpresheaves on $\mathcal{C}$.*

(ii) *For any class J of sieves $C \hookrightarrow y(X)$ on objects X of $\mathcal{C}$,*

$G_R \circ F_R(J) = \overline{J}$ is the topology generated by J,

i.e. the smallest topology containing J.

Furthermore, J and $\overline{J}$ define the same "closedness property" of subpresheaves and induce the same operation of closure of subpresheaves

$$(Q \hookrightarrow P) \longmapsto (\overline{Q} \hookrightarrow P)$$

which, in particular, is respected by base change along any $P' \to P$, in the sense that

$$\overline{Q \times_P P'} = \overline{Q} \times_P P' .$$

(iii) *For any class I of subpresheaves $Q \hookrightarrow P$ on $\mathcal{C}$,*

$F_R \circ G_R(I) = \overline{I}$ is the smallest "closure property" which contains I.

Furthermore, I and $\overline{I}$ define the same topology on $\mathcal{C}$

$$G_R(I) = G_R(\overline{I}) .$$

Any class of subpresheaves defines a topology:

- Consider a class I of subpresheaves $Q \hookrightarrow P$.

We need to verify that the class J of sieves $C \hookrightarrow y(X)$ such that

$$\begin{cases} \text{for any morphism } X' \xrightarrow{x} X \text{ and any } p \in P(X') \\ \text{one has } p \in Q(X') \text{ if } x'^*(p) \in Q(X'') \,,\ \forall\, (X'' \xrightarrow{x'} X') \in x^* C, \end{cases}$$

is a topology.

- As any intersection of topologies is a topology,

it is enough to consider the case where I has a unique element $Q \hookrightarrow P$.

- The above condition is verified by maximal sieves $C = y(X)$.
- By definition, this condition is stable under base change by any morphism $X' \xrightarrow{x} X$.
- Consider sieves $C \hookrightarrow y(X)$ and $C' \hookrightarrow y(X)$ such that

$C' \in J$ and $x^* C \in J$, $\forall\, (X' \xrightarrow{x} X) \in C'$.

As these properties are stable under base change,
it is enough to prove that any element $p \in P(X)$
such that $x^*(p) \in Q(X')$, $\forall\, (X' \xrightarrow{x} X) \in C$, is in $Q(X)$.

- For any $(X' \xrightarrow{x} X) \in C'$, we have $x^* C \in J$

and $x'^* \circ x^*(p) \in Q(X'')$, $\forall\, (X'' \xrightarrow{x'} X') \in x^* C$, which implies that $x^*(p) \in Q(X')$.
As $C' \in J$, we conclude that $p \in Q(X)$.
This means that $C \in J$.

- So J verifies (Trans) in addition to (Max) and (Stab).

Any class of sieves defines a "closedness property":

• Consider a class J of sieves and its image

$$I = F_R(J) = \left\{ Q \hookrightarrow P \;\middle|\; \begin{array}{l} \forall\,(C \hookrightarrow y(X)) \in J\,,\ \forall\,(X' \xrightarrow{x} X)\,, \\ \forall\, p \in P(X'),\ \text{one has } p \in Q(X') \\ \text{if } x'^*(p) \in Q(X'')\,,\ \forall\,(X'' \xrightarrow{x'} X') \in x^* C \end{array} \right\}.$$

• It is obvious from this definition that

$$\begin{cases} - & \text{all isomorphisms } Q \xrightarrow{\sim} P \text{ belong to } I, \\ - & \text{the class } I \text{ is respected by all base changes } P' \to P, \\ - & \text{it is also stable under intersections of elements } Q_k \hookrightarrow P,\ k \in K. \end{cases}$$

• We already know that $G_R \circ F_R(J) = \overline{J}$ is a topology and $\quad I = F_R(\overline{J})$.

• This implies that for any subpresheaf $Q \hookrightarrow P$
the smallest element of I which contains Q

$$\overline{Q} \lhook\joinrel\longrightarrow P$$

is characterized by the following formula at any object X of $\mathcal{C}$

$$\overline{Q}(X) = \{p \in P(X) \mid \exists\, C \in \overline{J}(X)\,,\ x^*(p) \in Q(X')\,,\ \forall\,(X' \xrightarrow{x} X) \in C\}.$$

• This formula implies that, for any morphism $P' \to P$,

$$\overline{Q \times_P P'} = \overline{Q} \times_P P' \text{ as subpresheaves of } P'.$$

• We conclude that $I = F_R(J) = F_R(\overline{J})$ is a "closedness property".

Topologies and "closedness properties" as fixed points:

• Consider a topology J and the associated "closedness property"

$$I = F_R(J)\,.$$

It defines a "closure operation" on subpresheaves

$$(Q \hookrightarrow P) \longmapsto (\overline{Q} \hookrightarrow P)$$

where, for any object X of $\mathcal{C}$,

$$\overline{Q}(X) = \{p \in P(X) \mid \exists\, C \in \overline{J}(X)\,,\ x^*(p) \in Q(X')\,,\ \forall\,(X' \xrightarrow{x} X) \in C\}.$$

If $C \hookrightarrow y(X)$ is a sieve belonging to $G_R \circ F_R(J) = G_R(I)$,
one has for any subpresheaf $Q \hookrightarrow P$ and any morphism $y(X) \to P$
the implication $C \subseteq Q \times_P y(X) \Rightarrow \overline{Q} \times_P y(X) = y(X)$.
This means that $\overline{C} = y(X)$ or, equivalently, $C \in J$.
We conclude that $J = G_R \circ F_R(J)$ is a fixed point.

• Consider a "closedness property" I and the associated topology $J = G_R(I)$.
A sieve $C \hookrightarrow y(X)$ belongs to J
if and only if, for any morphism $X' \to X$
and any $Q \hookrightarrow y(X')$ which belongs to I,
one has the implication $\quad x^*C \subseteq Q \Rightarrow Q = y(X')$.
This means that $\quad C \in J \quad$ if and only if $\quad \overline{C} = y(X)$.
We conclude that $\quad I = F_R \circ G_R(I) \quad$ is a fixed point.

III. Generation of topologies and provability:

- **A closed formula for the generation of topologies**
  - The dualities of sieves with presheaves and with subpresheaves.
  - Sieves and closedness properties of subpresheaves.
  - A generation formula based on closure operations.
  - Application to joins of topologies.
  - Application to finite products of toposes.
- **A generation formula in terms of multicoverings**
  - The notion of multicovering of an object.
  - Explicitation of closure operations of subpresheaves.
  - An explicit formula for generated topologies.
- **Topological interpretations of provability problems**
  - Topological interpretations of geometric axioms.
  - Reduction to atomic and Horn formulas.
  - Constructive interpretations of axioms in terms of covering sieves.
  - The problem of presentations of classifying toposes.
  - The case of presheaf type theories.
  - The case of cartesian theories.
  - The case of theories without functions symbols and without axioms.

Reminder on the duality of sieves and presheaves:

- For any essentially small category $\mathcal{C}$, there is a duality

  between $\quad T = \{(C \hookrightarrow y(X)) \mid X = \text{object of } \mathcal{C},\ C \hookrightarrow y(X) \text{ in } \widehat{\mathcal{C}}\}$

  and $\quad S = \{\text{presheaves } (P : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}) = \text{objects of } \widehat{\mathcal{C}}\}$

defined by the relation $\quad R \longhookrightarrow T \times S$

consisting in pairs $(C \hookrightarrow y(X), P)$ such that

$$\forall\, (X' \xrightarrow{x} X),\ P(X') = \mathrm{Hom}(y(X'), P) \longrightarrow \mathrm{Hom}(x^* C, P) \text{ is one-to-one.}$$

- This duality induces a pair of adjoint order preserving maps

$$\mathcal{P}(T) \underset{G_R}{\overset{F_R}{\rightleftarrows}} \mathcal{P}(S)$$

such that

- for any $J \subseteq T$, $F_R(J)$ is a subtopos of $\widehat{\mathcal{C}}$
  and $G_R \circ F_R(J)$ is the topology generated by J,
- for any $I \subseteq S$, $G_R(I)$ is a topology on $\mathcal{C}$
  and $F_R \circ G_R(I)$ is the subtopos generated by I.

- This induces a one-to-one correspondence

$$\left\{\frac{\text{topologies}}{\text{on } \mathcal{C}}\right\} \quad \underset{\underset{\sim}{G_R}}{\overset{\overset{F_R}{\sim}}{\rightleftarrows}} \quad \left\{\frac{\text{subtoposes}}{\text{of } \widehat{\mathcal{C}}}\right\}.$$

Reminder on the duality of sieves and subpresheaves:

- For any essentially small category $\mathcal{C}$, there is a duality
  between $\quad T = \{(C \hookrightarrow y(X)) \mid X = \text{object of } \mathcal{C},\ C = \text{sieve on } X\}$
  and $\quad S' = \{\text{monomorphisms } (Q \hookrightarrow P) \text{ in } \widehat{\mathcal{C}}\}$

defined by the relation $\quad R' \longhookrightarrow T \times S'$
consisting in pairs $(C \hookrightarrow y(X), Q \hookrightarrow P)$ such that

$$\forall\,(X' \xrightarrow{x} X),\ \forall\, p \in P(X'),\ [x'^*(p) \in Q(X''),\ \forall\,(X'' \xrightarrow{x'} X') \in x^*C] \Rightarrow p \in Q(X').$$

- This duality induces a pair of adjoint order preserving maps

$$\mathcal{P}(T) \underset{G_{R'}}{\overset{F_{R'}}{\rightleftarrows}} \mathcal{P}(S')$$

such that

- for any $J \subseteq T$, $F_{R'}(J)$ is a closedness property
  and $G_{R'} \circ F_{R'}(J)$ is the topology generated by J,
- for any $I \subseteq S'$, $G_{R'}(I)$ is a topology on $\mathcal{C}$
  and $F_{R'} \circ G_{R'}(I)$ is the closedness property generated by I.

- This induces a one-to-one correspondence

$$\left\{\frac{\text{topologies}}{\text{on } \mathcal{C}}\right\} \underset{\underset{\sim}{G_{R'}}}{\overset{\overset{F_{R'}}{\sim}}{\rightleftarrows}} \left\{\frac{\text{closedness properties of}}{\text{subpresheaves } Q \hookrightarrow P}\right\}.$$

Reminder on topologies and closedness properties:

Definition. – *Any* $J \subseteq T = \{(C \hookrightarrow y(X)) \mid X = \text{object of } \mathcal{C},\ C = \text{sieve on } X\}$ *is a* *topology* *if and only if*

$$\begin{cases} \bullet & J \text{ contains maximal sieves } J = y(X) \overset{=}{\hookrightarrow} y(X), \\ \bullet & J \text{ is stable by pull-backs along morphisms } X' \xrightarrow{x} X, \\ \bullet & \text{a sieve } C \hookrightarrow y(X) \text{ belongs to } J \text{ if there exists } (C' \hookrightarrow y(X)) \in J \\ & \text{such that } (x^*C \hookrightarrow y(X')) \in J, \forall\, (X' \xrightarrow{x} X) \in C'. \end{cases}$$

Definition. –
A property of subpresheaves $I \subseteq S' = \{(Q \hookrightarrow P) = \text{monomorphism of } \widehat{\mathcal{C}}\}$ *is a "closedness property" if and only if:*

$$\begin{cases} \bullet & \text{isomorphisms } Q \xrightarrow{\sim} P \text{ belong to } I, \\ \bullet & I \text{ is stable by pull-backs along morphisms } P' \to P \text{ of } \widehat{\mathcal{C}}, \\ \bullet & I \text{ is stable under intersections in the sense that} \\ & \qquad (Q_k \hookrightarrow P) \in I,\ \forall k \in K \Rightarrow \left(\bigcap\limits_{k \in K} Q_k \hookrightarrow P\right) \in I, \\ \bullet & \text{if, for any } Q \hookrightarrow P \text{ in } \widehat{\mathcal{C}}, \\ & \overline{Q} \hookrightarrow P \text{ denotes the smallest element of } I \text{ containing } Q, \\ & \text{we have for any morphism } P' \to P \qquad \overline{P' \times_P Q} = P' \times_P \overline{Q}. \end{cases}$$

Sieves and covering presieves:

• A sieve on an object X of $\mathcal{C}$ is a subobject

$$C \hookrightarrow y(X) \qquad \text{in} \quad \widehat{\mathcal{C}}$$

or, equivalently, a collection of morphisms $\quad (X' \xrightarrow{x} X)$

such that, for any morphism $X'' \xrightarrow{x'} X'$,

$$(X' \xrightarrow{x} X) \in C \Rightarrow (x \circ x' : X'' \to X' \to X) \in C.$$

Definition. – *A presieve on an object X of $\mathcal{C}$ is a family of morphisms*

$$(x_i : X_i \longrightarrow X)_{i \in I}\,.$$

Remarks:

- Any such presieve generates a sieve which is

  $$\{X' \xrightarrow{x} X \mid x \text{ factorizes though at least some } X_i \to X,\, i \in I\}.$$

- Any sieve is generated by presieves.

Definition. –
Let $\quad J \subseteq T = \{(C \hookrightarrow y(X)) \mid \text{sieves } C \text{ on objects } X \text{ of } \mathcal{C}\}$ be a topology
or, more generally, a family of sieves stable under pull-backs along all $X' \xrightarrow{x} X$.
Then a presieve $(X_i \xrightarrow{x_i} X)_{i \in I}$ is called J-covering if and only if
its generated sieve contains some $\quad (C \hookrightarrow y(X)) \in J$.

Stabilisation of families of sieves:

Definition. – *A family of sieves* $J \subseteq T = \{C \hookrightarrow y(X)\}$ *will be called "stable" if it is respected by pull-backs along morphisms* $X' \xrightarrow{x} X$ *of* $\mathcal{C}$.

Lemma. – *Any family of sieves* $J \subseteq T = \{C \hookrightarrow y(X)\}$ *generates a stable family which is*

$$J_s = \{C' \hookrightarrow y(X') \mid \exists\, (X' \xrightarrow{x} X)\,,\ \exists\, (C \hookrightarrow y(X)) \in J\,,\ C' = x^* C\}.$$

Remarks:

- One has the inclusions $J \subseteq J_s \subseteq \overline{J} =$ topology generated by J and they define
  - the same subtopos $F_R(J) = F_R(J_s) = F_R(\overline{J})$,
  - the same closedness property $F_{R'}(J) = F_{R'}(J_s) = F_{R'}(\overline{J})$.
- A subpresheaf $Q \longleftrightarrow P$ is J-closed, or J_s-closed, or $\overline{J}$-closed if and only if, for any $p \in P(X)$ and any $(C \hookrightarrow y(X)) \in J_s$,
$$x^*(p) \in Q(X')\,,\ \forall\, (X' \xrightarrow{x} X) \in C \Rightarrow p \in Q(X).$$
- The family J induces a notion of J_s-covering presieves.

A closed formula for generated topologies:

Theorem (O.C., L.L., see [Engendrement] improving a formula of [TST]). – *Let $J \subseteq T = \{C \hookrightarrow y(X)\}$ be a class of sieves on objects X of $\mathcal{C}$.*
Let J_s be the stabilisation of J

$$J_s = \{C' \hookrightarrow y(X') \mid \exists\,(X' \xrightarrow{x} X)\,,\ \exists\,(C \hookrightarrow y(X)) \in J\,,\ C' = x^*C\}.$$

Let $\overline{J}$ be the topology on $\mathcal{C}$ generated by J or J_s. Then a sieve on an object X of $\mathcal{C}$, $C \hookrightarrow y(X)$ belongs to $\overline{J}$ if and only if any sieve $C' \hookrightarrow y(X)$ such that

- *C' contains C,*
- *C' is J_s-closed in the sense that an arbitrary morphism $x : X' \longrightarrow X$ belongs to C' if the sieve on X'*

  $$\{X'' \xrightarrow{x'} X' \mid (x \circ x' : X'' \to X) \in C'\}$$

  contains an element of J_s,

is the maximal sieve $y(X) \overset{=}{\hookrightarrow} y(X)$.

Proof: We already know that J, J_s and $\overline{J}$ define
the same "closedness property" on subpresheaves $Q \hookrightarrow P$
and so the same "closure operation" $(Q \hookrightarrow P) \mapsto (\overline{Q} \hookrightarrow P)$.

The theorem statement means that $C \hookrightarrow y(X)$ belongs to $\overline{J}$
if and only if $\overline{C} = y(X)$.

Application to joins of topologies:

Corollary. –
Let $(J_k)_{k\in K}$ be a family of topologies on $\mathcal{C}$.
Let $J = \bigvee_{k\in K} J_k$
be the smallest topology which contains all J_k's, $k \in K$.
Then a sieve on a object X of $\mathcal{C}$ $C \hookrightarrow y(X)$
belongs to J if and only if any sieve $C' \hookrightarrow y(X)$ such that

$$\begin{cases} \bullet \quad C' \text{ contains } C, \\ \bullet \quad C' \text{ is } J_k\text{-closed for any } k \in K, \end{cases}$$

is the maximal sieve $y(X) \xhookrightarrow{=} y(X)$.

Proof:

- Indeed, the class J_s of sieves $C \hookrightarrow y(X)$ defined as the union of the classes J_k, $k \in K$, is stable under pull-backs along morphisms $X' \xrightarrow{x} X$ of $\mathcal{C}$. By definition, it generates the topology J.
- To conclude, we observe that a sieve $C' \hookrightarrow y(X)$ is J_s-closed if and only if it is J_k-closed for any $k \in K$.

Application to the construction of finite products of toposes:

Theorem. –
Consider topologies $J_1, \cdots, J_n$ on essentially small categories $\mathcal{C}_1, \cdots, \mathcal{C}_n$.
Consider the product category $\quad \mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$
endowed with the induced topologies $J_1, \cdots, J_n$.
Then the product topos in the 2-category of toposes

$$\mathcal{E} = \widehat{(\mathcal{C}_1)}_{J_1} \times \cdots \times \widehat{(\mathcal{C}_n)}_{J_n}$$

can be constructed as the topos of sheaves on the product category

$$\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$$

endowed with the topology J for wich
a sieve $\quad C \hookrightarrow y(X_1 \times \cdots \times X_n) \quad$ *belongs to J*
if and only if any sieve $C' \hookrightarrow y(X_1 \times \cdots \times X_n)$ such that

$$\begin{cases} \bullet \quad C' \text{ contains } C, \\ \bullet \quad C' \text{ is } J_k\text{-closed for any } k \in K, \end{cases}$$

is the maximal sieve $y(X_1 \times \cdots \times X_n) \xrightarrow{=} y(X_1 \times \cdots \times X_n)$.

This theorem is a consequence of the previous theorem and:

Proposition. – *Let $\mathcal{C}$ and $\mathcal{D}$ be essentially small categories. Then the presheaf topos $\widehat{\mathcal{C} \times \mathcal{D}}$ is a product of the presheaf toposes on $\mathcal{C}$ and $\mathcal{D}$, $\quad \widehat{\mathcal{C}} \times \widehat{\mathcal{D}}$.*

Products of toposes and products of topological spaces:

• To any topological space X are associated

$$\begin{cases} - & \text{the category } \mathcal{C}_X \text{ of open subsets of } X, \\ - & \text{the topology } J_X \text{ on } \mathcal{C}_X \text{ defined by the usual notion of covering,} \\ - & \text{the topos } \mathcal{E}_X = \widehat{(\mathcal{C}_X)}_{J_X} \text{ of sheaves on } X. \end{cases}$$

This defines a functor $\{\text{category of topological spaces}\} \longrightarrow \{\text{category of toposes}\}$.

• In particular, any topological spaces $X_1, \cdots, X_n$ define a topos morphism

$$\mathcal{E}_{X_1 \times \cdots \times X_n} \longrightarrow \mathcal{E}_{X_1} \times \cdots \times \mathcal{E}_{X_n}.$$

Proposition. –
For the natural morphism $\mathcal{E}_{X_1 \times \cdots \times X_n} \longrightarrow \mathcal{E}_{X_1} \times \cdots \times \mathcal{E}_{X_n}$ to be an isomorphism, it suffices that all factors X_i's, except possibly one, are locally compact.

Remark: If $\mathcal{E}_{X_1 \times \cdots \times X_n} \longrightarrow \mathcal{E}_{X_1} \times \cdots \times \mathcal{E}_{X_n}$ is an isomorphism of toposes,
the topos $\mathcal{E}_{X_1 \times \cdots \times X_n}$ of sheaves on $X_1 \times \cdots \times X_n$
can be constructed as the topos of sheaves on the product category $\mathcal{C}_{X_1} \times \cdots \times \mathcal{C}_{X_n}$
endowed with the topology $J = J_{X_1} \vee \cdots \vee J_{X_n}$
for which a sieve $C \hookrightarrow y(U_1 \times \cdots \times U_n)$ belongs to J
if and only if any sieve $C' \hookrightarrow y(U_1 \times \cdots \times U_n)$ such that

$$\begin{cases} \bullet & C' \text{ contains } C, \\ \bullet & C' \text{ is } J_{X_i}\text{-closed for any } i,\ 1 \leq i \leq n, \end{cases}$$

is the maximal sieve on $U_1 \times \cdots \times U_n$.

Multicoverings:

- Let J be a class of sieves $C \hookrightarrow y(X)$ of objects X of $\mathcal{C}$.
  Let J_s be the "stabilisation" of J.
- A J_s-covering of an object X is a presieve $(x_i : X_i \longrightarrow X)_{i \in I}$ whose generated sieve is maximal or contains an element of J_s.

Definition. – *A J_s-multicovering of an object X of $\mathcal{C}$ is a sequence*

$$\cdots \longrightarrow \mathfrak{X}_n \xrightarrow{f_n} \mathfrak{X}_{n-1} \longrightarrow \cdots \xrightarrow{f_2} \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0$$

where

- *each $\mathfrak{X}_k$, $k \in \mathbb{N}$, is a set of morphisms of $\mathcal{C}$,*
- *all morphisms in $\mathfrak{X}_0$ have target X and make up a J_s-covering of X,*
- *for any $n \geq 1$ and $x_n \in X_n$,*
  the target of x_n is the source of $f_n(x_n) \in \mathfrak{X}_{n-1}$,
- *for any $n \geq 1$ and $x_{n-1} \in X_{n-1}$, the fiber*
  $$\{x_n \in \mathfrak{X}_n \mid f_n(x_n) = x_{n-1}\}$$
  is empty or makes up a J_s-covering of the source of x_{n-1},
- *there is no infinite sequence $x_n \in X_n\,,\ n \in \mathbb{N}$,*
  such that $f_n(x_n) = x_{n-1}, \forall\, n \geq 1$.

Explicitation of the operation of closure of subpresheaves:

- Let J be a class of sieves on $\mathcal{C}$, J_s its "stabilisation" and $\overline{J}$ the generated topology.
- We know that J, J_s and $\overline{J}$ define the same "closedness property" of subpresheaves and the same operation of closure $(Q \hookrightarrow P) \mapsto (\overline{Q} \hookrightarrow P)$.

Theorem (O.C., L.L., to appear in [Engendrement]). –
Consider a subpresheaf $Q \hookrightarrow P$ of a presheaf P on $\mathcal{C}$.
Let $\overline{Q} \hookrightarrow P$ be its closure with respect to J, J_s or $\overline{J}$.
Then an element $p \in P(X)$
belongs to $\overline{Q}(X)$
if and only if there exists a J_s-multicovering of X

$$\cdots \longrightarrow \mathfrak{X}_n \xrightarrow{f_n} \mathfrak{X}_{n-1} \longrightarrow \cdots \xrightarrow{f_2} \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0$$

such that, for any $n \in \mathbb{N}$ and $x_n \in X_n$, we have

- *either x_n belongs to the image of $\mathfrak{X}_{n+1} \xrightarrow{f_{n+1}} \mathfrak{X}_n$,*
- *or the empty sieve on the source of x_n is J_s-covering,*
- *or, denoting $x_{n-1} = f_n(x_n)$, $x_{n-2} = f_{n-1}(x_{n-1}), \cdots, x_0 = f_1(x_1)$, the composite $x_0 \circ x_1 \circ \cdots \circ x_n : X_n \longrightarrow X$ verifies the property $(x_0 \circ x_1 \circ \cdots \circ x_n)^*(p) \in Q(X_n)$.*

Verification of stability under pull-backs:

- Consider as before J, J_s and $\overline{J}$.
  Consider a subpresheaf $Q \hookrightarrow P$.
- For any object X, let $\widetilde{Q}(X) \subseteq P(X)$
  be the subset of elements of P which, as in the theorem, can be sent into $Q \hookrightarrow P$ by some J_s-multicovering.
- We first have to check that any morphism $x : X' \to X$
  sends $\quad \widetilde{Q}(X) \subseteq P(X) \quad$ into $\quad \widetilde{Q}(X') \subseteq P(X')$.
- Given $p \in \widetilde{Q}(X)$ and an adapted J_s-multicovering of X

$$\cdots \longrightarrow \mathfrak{X}_n \xrightarrow{f_n} \mathfrak{X}_{n-1} \longrightarrow \cdots \xrightarrow{f_2} \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0,$$

  it is enough to construct a J_s-multicovering of X'
  as part of a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathfrak{X}'_n \arrow[r, "f'_n"] \arrow[d] & \mathfrak{X}'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r, "f'_2"] & \mathfrak{X}'_1 \arrow[r, "f'_1"] \arrow[d] & \mathfrak{X}'_0 \arrow[d] \\
\cdots \arrow[r] & \mathfrak{X}_n \arrow[r, "f_n"] & \mathfrak{X}_{n-1} \arrow[r] & \cdots \arrow[r, "f_2"] & \mathfrak{X}_1 \arrow[r] & \mathfrak{X}_0
\end{tikzcd}$$

such that

- for any $x_n' \in \mathfrak{X}_n'$ of image $x_n \in \mathfrak{X}_n$, there is an associated morphism

$$\text{source}\,(x_n') \xrightarrow{t_{x_n'}} \text{source}\,(x_n)$$

- for any $x_n' \in \mathfrak{X}_n'$ and its images $x_n \in \mathfrak{X}_n$, $x_{n-1}' \in \mathfrak{X}_{n-1}'$, $x_{n-1} \in \mathfrak{X}_{n-1}$, the square

$$\begin{tikzcd} \bullet \arrow[r, "x_n'"] \arrow[d, "t_{x_n'}"'] & \bullet \arrow[d, "t_{x_{n-1}'}"] \\ \bullet \arrow[r, "x_n"] & \bullet \end{tikzcd} \quad \left(\text{or} \quad \begin{tikzcd} \bullet \arrow[r, "x_0'"] \arrow[d, "t_{x_0'}"'] & X' \arrow[d, "x"] \\ \bullet \arrow[r, "x_0"] & X \end{tikzcd} \right)$$

is commutative.

Verification of the closedness property:

- We have to verify that the subpresheaf
$$\widetilde{Q} \hookrightarrow P \qquad \text{is closed.}$$
- Consider an element $p \in P(X)$
such that there exists a J_s-covering $(X_k \xrightarrow{x_k} X)_{k \in K}$
verifying $\qquad x_k^*(p) \in \widetilde{Q}(X_k)\,,\ \forall\, k \in K.$
- By definition of $\widetilde{Q}$, each X_k has a J_s-multicovering
$$\cdots \longrightarrow \mathfrak{X}_n^k \xrightarrow{f_n^k} \mathfrak{X}_{n-1}^k \longrightarrow \cdots \longrightarrow \mathfrak{X}_1^k \xrightarrow{f_1^k} \mathfrak{X}_0^k$$
which allows to send $x_k^*(p)$ into $Q \hookrightarrow P$.
- Then the formulas
$$\mathfrak{X}_0 = \left\{ (X_k \xrightarrow{x_k} X) \mid k \in K \right\}$$
and
$$\mathfrak{X}_n = \coprod_{k \in K} \mathfrak{X}_{n-1}^k \quad \text{for} \quad n \geq 1$$
define a J_s-multicovering of X
which sends $p \in P(X)$ into $Q \hookrightarrow P$.
- This means that $\qquad p \in \widetilde{Q}(X)$.

Verification of minimality:

- Consider a subpresheaf $Q' \hookrightarrow P$
  which is closed with respect to J, J_s or $\overline{J}$ and which contains $Q \hookrightarrow P$.
  We have to check that Q' contains $\widetilde{Q} \hookrightarrow P$.
- Consider an element $p \in \widetilde{Q}(X)$ and a J_s-multicovering of X
  $$\cdots \longrightarrow \mathfrak{X}_n \xrightarrow{f_n} \mathfrak{X}_{n-1} \longrightarrow \cdots \longrightarrow \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0$$
  which sends p into Q.
- For any $n \geq 0$, let $\mathfrak{X}'_n \subseteq \mathfrak{X}_n$ be the subset of elements x_n
  whose associated branch $x_n, f_n(x_n) = x_{n-1}, \cdots, f_1(x_1) = x_0$
  verifies $(x_0 \circ x_1 \circ \cdots \circ x_n)^*(p) \notin Q'(X)$.
- We have to prove that all $\mathfrak{X}'_n$, $n \geq 0$, are empty.
- If they were not all empty, there would exist
  $$n \geq 0 \quad \text{and} \quad x_n \in \mathfrak{X}_n \quad \text{such that}$$
  $$\{x_{n+1} \in \mathfrak{X}'_{n+1} \mid f_{n+1}(x_{n+1}) = x_n\} = \emptyset.$$
  This would yield a contradiction as
  $$\begin{cases} - \quad \text{either } \{x_{n+1} \in \mathfrak{X}_{n+1} \mid f_{n+1}(x_{n+1}) = x_n\} \text{ is } J_s\text{-covering,} \\ - \quad \text{or } (x_0 \circ x_1 \circ \cdots \circ x_n)^*(p) \in Q(X). \end{cases}$$

An explicit formula for generated topologies:

• Let J be a class of sieves $C \hookrightarrow y(X)$ on $\mathcal{C}$, J_s be its "stabilisation" and $\overline{J}$ their generated topology.

Corollary. –
A sieve on an object X

$$C \hookrightarrow y(X)$$

belongs to the generated topology $\overline{J}$ if and only if there exists a J_s-multicovering of X

$$\cdots \longrightarrow \mathfrak{X}_n \xrightarrow{f_n} \mathfrak{X}_{n-1} \longrightarrow \cdots \longrightarrow \mathfrak{X}_1 \xrightarrow{f_1} \mathfrak{X}_0$$

such that, for any $n \in \mathbb{N}$ and $x_n \in \mathfrak{X}_n$, we have

- *either x_n belongs to the image of $\mathfrak{X}_{n+1} \xrightarrow{f_{n+1}} \mathfrak{X}_n$,*
- *or the empty sieve on the source of x_n is J_s-covering,*
- *or, denoting*

$$f_n(x_n) = x_{n-1}, \cdots, f_1(x_1) = x_0\,,$$

the composite

$$x_0 \circ x_1 \circ \cdots \circ x_n : X_n \longrightarrow X$$

is an element of C.

Topological interpretations of geometric axioms:

- Consider a geometric first-order theory $\mathbb{T}$ in a vocabulary (or "signature") Σ consisting in

$$\begin{cases} - & \text{object names (or "sorts") } A_i, \\ - & \text{morphism names (or "function symbols") } f : A_1 \cdots A_n \to A, \\ - & \text{subobject names (or "relation symbols") } R \rightarrowtail A_1 \cdots A_n. \end{cases}$$

Reminder. – *For any model M of such a geometric theory $\mathbb{T}$ in a topos $\mathcal{E}$, corresponding to a topos morphism $\mathcal{E} \xrightarrow{m} \mathcal{E}_{\mathbb{T}}$, we have:*

(i) *Any sort A_i interprets as an object MA_i of $\mathcal{E}$.*

(ii) *Any geometric formula $\varphi(x_1^{A_1} \cdots x_n^{A_n})$ of Σ interprets as a subobject*

$$M\varphi(x_1^{A_1} \cdots x_n^{A_n}) \hookrightarrow MA_1 \times \cdots \times MA_n.$$

(iii) *Any implication (or "sequent") $\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$ interprets as an embedding of subobjects of $MA_1 \times \cdots \times MA_n$*

$$M(\varphi \wedge \psi)(x_1^{A_1} \cdots x_n^{A_n}) \hookrightarrow M\varphi(x_1^{A_1} \cdots x_n^{A_n})$$

which is an epimorphism (and so an isomorphism) if and only if M verifies $\varphi \vdash \psi$.

Reduction from general geometric formulas to Horn formulas:

Definition. –
Let Σ be a first-order vocabulary (or "signature").

(i) *A geometric formula $\varphi(\vec{x})$ of Σ is called "atomic" if it is deduced from relation or equality formulas*

$$R(x_1^{A_1}, \cdots, x_n^{A_n}) \quad \text{or} \quad x_1^A = x_2^A$$

by replacing finitely many times variables by morphism formulas

$$x^A = f(x_1^{B_1}, \cdots, x_m^{B_m}) \quad \text{for} \quad f : B_1 \cdots B_m \to A \text{ in } \Sigma.$$

(ii) *A geometric formula $\varphi(\vec{x})$ of Σ is called Horn if it is a finite conjunction of atomic formulas $\varphi_i(\vec{x})$*

$$\varphi(\vec{x}) = \varphi_1(\vec{x}) \wedge \cdots \wedge \varphi_k(\vec{x}) \,.$$

Lemma. –
Any geometric formula $\varphi(\vec{x})$ can be written in equivalent form

$$\varphi(\vec{x}) = \bigvee_{i \in I} \exists\, (\vec{x}_i) \varphi_i(\vec{x}_i, \vec{x})$$

where each $\varphi_i(\vec{x}_i, \vec{x})$ is a Horn formula.

Reduction to topological interpretations of Horn formulas:

Corollary. –
Let $\mathbb{T}$ be a geometric first-order theory in a vocabulary Σ.
Then it is possible to associate to any geometric sequent of Σ

$$\varphi(\vec{x}) \vdash \psi(\vec{x})$$

a double family $\mathfrak{X}_{\vec{x},\varphi,\psi}$ consisting in

- *a family of Horn formulas*
$$\varphi_i(\vec{x}_i)\,, \quad i \in I\,,$$
- *for each index i, a family of Horn formulas*
$$\varphi_{i,j}(\vec{x}_i, \vec{x}_{i,j})\,, \quad j \in I_i\,,$$

such that, for any model M of $\mathbb{T}$ in a topos $\mathcal{E}$,
the implication $\varphi(\vec{x}) \vdash \psi(\vec{x})$ is verified by M
if and only if

- *for any index $i \in I$, the family of projections in $\mathcal{E}$*
$$M(\varphi_{i,j}(\vec{x}_i, \vec{x}_{i,j}) \wedge \varphi_i(\vec{x}_i)) \longrightarrow M\varphi_i(\vec{x}_i)$$
is globally epimorphic.

Concrete reduction of geometric axioms to topology generation:

Proposition. – *Let $\mathbb{T}$ be a geometric first-order theory in a vocabulary Σ. Suppose that the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ is presented as*

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J \quad \text{through} \quad \ell : \mathcal{C} \to \mathcal{E}_{\mathbb{T}}$$

where

- $\mathcal{C}$ *has arbitrary finite limits, i.e. finite products and fiber products,*
- *any "sort" A of Σ interprets as an object UA of $\mathcal{C}$,*
- *any "function symbol" $f : A_1 \cdots A_n \to A$ of Σ interprets as a morphism of $\mathcal{C}$* $\quad Uf : UA_1 \times \cdots \times UA_n \to UA,$
- *any "relation symbol" $R \rightarrowtail A_1 \cdots A_n$ of Σ interprets as a subobject of $\mathcal{C}$* $\quad UR \hookrightarrow UA_1 \times \cdots \times UA_n,$

so that any Horn formula $\varphi(x_1^{A_1} \cdots x_n^{A_n})$ of Σ interprets as a subobject of $\mathcal{C}$ $\quad U\varphi \hookrightarrow UA_1 \times \cdots \times UA_n.$

Then a geometric sequent $\varphi(\vec{x}) \vdash \psi(\vec{x})$ of Σ is provable in a quotient theory $\mathbb{T}'$ of $\mathbb{T}$ corresponding to a topology $J' \supseteq J$ if and only if the associate families of projection morphisms

$$(U(\varphi_{i,j}(\vec{x}_i, \vec{x}_{i,j}) \wedge \varphi_i(\vec{x}_i)) \longrightarrow U\varphi_i(\vec{x}_i))_{j \in I_i}, \ i \in I,$$

defined by the double family of Horn formulas $\mathfrak{X}_{\vec{x},\varphi,\psi}$ are J'-coverings.

The problem of presentations of classifying toposes:

Problem. – *Given a geometric first-order theory* $\mathbb{T}$ *in a vocabulary* Σ, *how to present its classifying topos in terms of a site* $(\mathcal{C}, J)$

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J$$

such that

$$\begin{cases} \bullet & \mathcal{C} \text{ has arbitrary finite limits,} \\ \bullet & \text{elements of the vocabulary } \Sigma \text{ interpret in } \mathcal{C}. \end{cases}$$

Hints:

- One may take
  $\begin{cases} \mathcal{C} = \mathcal{C}_{\mathbb{T}} \text{ (syntactic category of } \mathbb{T}\text{),} \\ J = J_{\mathbb{T}} \text{ (syntactic topology on } \mathcal{C}_{\mathbb{T}}\text{).} \end{cases}$
- More generally, one may write
  $\mathbb{T} =$ quotient of a theory $\mathbb{T}_0$ in the same vocabulary Σ, and take
  $\begin{cases} \mathcal{C} = \mathcal{C}_{\mathbb{T}_0} \text{ (syntactic category of } \mathbb{T}_0\text{),} \\ J = \text{topology on } \mathcal{C}_{\mathbb{T}_0} \text{ generated by } J_{\mathbb{T}_0} \\ \text{and the covering families associated with the axioms of } \mathbb{T}. \end{cases}$
- Even more generally, one can first replace $\mathbb{T}$ by
  $\mathbb{T}' =$ geometric first-order theory in a vocabulary Σ'
  which is "syntactically equivalent" in the sense that $\mathcal{C}_{\mathbb{T}} \cong \mathcal{C}_{\mathbb{T}'}$.

The case of presheaf type theories:

Definition. – *A geometric first-order theory $\mathbb{T}$ in a vocabulary Σ is called "presheaf type" if its classifying topos is a topos of presheaves $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$ on some category $\mathcal{C}$.*

Examples:

- Any theory $\mathbb{T}$ consisting in a vocabulary Σ without axioms is presheaf type.
- More generally, any "cartesian theory" is presheaf type.
- In particular, any "algebraic" or "Horn" theory is presheaf type.

Theorem (O.C., see [TST]). – *For any presheaf type theory $\mathbb{T}$, one has*

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$$

for $\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} \cong (\mathbb{T}\text{-mod}\,(\mathrm{Set}))_{\mathrm{ft}}^{\mathrm{op}}$ *where*

- $\mathcal{C}_{\mathbb{T}}^{\mathrm{ir}}$ *is the full subcategory of $\mathcal{C}_{\mathbb{T}}$ on objects which are "irreducible" in the sense that their only $J_{\mathbb{T}}$-covering sieve is the maximal sieve.*
- $\mathbb{T}\text{-mod}\,(\mathrm{Set})_{\mathrm{fp}}$ *is the full subcategory of $\mathbb{T}\text{-mod}\,(\mathrm{Set})$ on set-valued models of $\mathbb{T}$ which are "finitely presentable" by geometric formulas.*

The case of cartesian theories:

Theorem. –
If $\mathbb{T}$ is a "cartesian" theory, it is presheaf type and one can write

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$$

with $\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{\mathrm{ir}} = \mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$
where $\mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$ *is the "syntactic cartesian theory" of* $\mathbb{T}$ *consisting in*

- *objects which are "cartesian formulas"*
  in the vocabulary Σ *of* $\mathbb{T}$*, meaning formulas of the form*
  $$(\exists\, \vec{y})\ \varphi(\vec{x}, \vec{y})$$
  where $\varphi(\vec{x}, \vec{y})$ *is a Horn formula*
  and the sequent
  $$\varphi(\vec{x}, \vec{y}) \wedge \varphi(\vec{x}, \vec{y}\,') \vdash \vec{y} = \vec{y}\,' \qquad \textit{is provable in } \mathbb{T},$$
- *morphisms which are "cartesian formulas"* $\theta(\vec{x}, \vec{y})$
  $$\varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{y})} \psi(\vec{y})$$
  which are $\mathbb{T}$*-provably functional.*

Reduction to theories without function symbols:

Lemma. – *For any geometric first-order theory $\mathbb{T}$ in a vocabulary Σ, there is a syntactically equivalent geometric theory $\mathbb{T}'$ whose vocabulary Σ' does not contain function symbols.*

Remark: The meaning of “syntactically equivalent” is that the syntactic categories $\mathcal{C}_{\mathbb{T}}$ and $\mathcal{C}_{\mathbb{T}'}$ of $\mathbb{T}$ and $\mathbb{T}'$ are equivalent:

$$\mathcal{C}_{\mathbb{T}} \cong \mathcal{C}_{\mathbb{T}'},$$

implying

$$\mathcal{E}_{\mathbb{T}} \cong \mathcal{E}_{\mathbb{T}'}.$$

Proof:

- Replace each function symbol $f : A_1 \cdots A_n \to A$ of Σ by a relation symbol $R_f \hookrightarrow A_1 \cdots A_n A$ completed by the axioms

$$\begin{cases} R_f(x_1^{A_1}, \cdots, x_n^{A_n}, y^A) \wedge R_f(x_1^{A_1}, \cdots, x_n^{A_n}, z^A) \vdash y^A = z^A, \\ \top \vdash_{x_1^{A_1}, \cdots, x_n^{A_n}} (\exists\, y^A)\, R_f(x_1^{A_1}, \cdots, x_n^{A_n}, y^A)\,. \end{cases}$$

- Then replacing each substitution of variables $y^A = f(x_1^{A_1}, \cdots, x_n^{A_n})$ by $R_f(x_1^{A_1}, \cdots, x_n^{A_n}, y^A)$ and each relation $R_f(x_1^{A_1}, \cdots, x_n^{A_n}, y^A)$ by the equality $f(x_1^{A_1}, \cdots, x_n^{A_n}) = y^A$ defines an equivalence of categories $\mathcal{C}_{\mathbb{T}} \cong \mathcal{C}_{\mathbb{T}'}$.

The case of theories without function symbols and without axioms:

In that case, the cartesian syntactic category and the classifying topos can be described fully explicitly:

Proposition. – *Let Σ be a vocabulary without function symbols.*

Then one can write $\quad \mathcal{E}_\Sigma \cong \widehat{\mathcal{C}}$

where $\mathcal{C} = \mathcal{C}_\Sigma^{\mathrm{cart}}$ is the syntactic cartesian category of Σ explicited as follows:

(1) *The objects of $\mathcal{C} = \mathcal{C}_\Sigma^{\mathrm{cart}}$ are finite conjunctions*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) = \bigwedge_{1 \leq k \leq \ell} \varphi_k(x_1^{A_1}, \cdots, x_n^{A_n})$$

of atomic formulas $\quad \varphi_k(x_1^{A_1}, \cdots, x_n^{A_n})$

which are relation symbols $\quad R(x_{i_1}^{A_{i_1}}, \cdots, x_{i_m}^{A_{i_m}})$

or equality relations $\quad x_{i_1}^{A_{i_1}} = x_{i_2}^{A_{i_2}} = \cdots = x_{i_m}^{A_{i_m}}$

in part of the variables $\quad x_1^{A_1}, \cdots, x_n^{A_n}$.

(2) *The morphisms of $\mathcal{C} = \mathcal{C}_\Sigma^{\mathrm{cart}}$*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \longrightarrow \psi(x_{\alpha_1}^{A_{\alpha_1}}, \cdots, x_{\alpha_{n'}}^{A_{\alpha_{n'}}})$$

are projections associated with injective maps

$$\alpha : \{1, \cdots, n'\} \hookrightarrow \{1, \cdots, n\}$$

which transform all atomic components of ψ into atomic components of φ.

IV. Geometric operations on subtoposes

- **Unions, intersections and differences of subtoposes**
  - Topological expressions.
  - Logical expressions.
- **Existential push-forward and pull-back of subtoposes**
  - Logical expression of push-forward.
  - Semantic expression of pull-back.
  - Topological expression of push-forward and pull-back.
  - Actions of correspondences and their topological expression.
- **Compatibility of pull-backs with unions of subtoposes**
  - The case of locally connected morphisms and its consequences.
  - Fibrations, Giraud topologies and locally connected morphisms.
  - Factorizations of topos morphisms through locally connected morphisms.
  - Galois correspondences associated with essential morphisms of toposes.
  - Characterization of pull-backs under essential morphisms.
  - Characterization of pull-backs under locally connected morphisms.

Unions, intersections and differences of toposes:

Proposition. – *Let $\mathcal{E}$ be a topos.*

(i) *Any family of subtoposes $(\mathcal{E}_i \hookrightarrow \mathcal{E})_{i\in I}$ has a union $\bigvee\limits_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ and an intersection $\bigwedge\limits_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ characterized by the properties that, for any subtopos $\mathcal{E}' \hookrightarrow \mathcal{E}$,*

$$\bigvee_{i\in I} \mathcal{E}_i \subseteq \mathcal{E}' \Leftrightarrow \mathcal{E}_i \subseteq \mathcal{E}' , \quad \forall\, i \in I ,$$

$$\mathcal{E}' \subseteq \bigwedge_{i\in I} \mathcal{E}_i \Leftrightarrow \mathcal{E}' \subseteq \mathcal{E}_i , \quad \forall\, i \in I .$$

(ii) *For any subtoposes $\mathcal{E}_1, \mathcal{E}_2$ of $\mathcal{E}$, there exists a subtopos $\mathcal{E}_1 \backslash \mathcal{E}_2 \hookrightarrow \mathcal{E}$ characterized by the property that, for any $\mathcal{E}' \hookrightarrow \mathcal{E}$,*

$$\mathcal{E}_1 \backslash \mathcal{E}_2 \subseteq \mathcal{E}' \Leftrightarrow \mathcal{E}_1 \subseteq \mathcal{E}_2 \vee \mathcal{E}' .$$

Remark: (ii) means that the functor $\mathcal{E}' \mapsto \mathcal{E}_2 \vee \mathcal{E}'$ has a left-adjoint $\mathcal{E}_1 \mapsto \mathcal{E}_1 \backslash \mathcal{E}_2$.

Corollary. –

(i) *The functor $\mathcal{E}' \mapsto \mathcal{E}_2 \vee \mathcal{E}'$ respects arbitrary intersections.*

(ii) *As a formal consequence, intersection functors $\mathcal{E}' \mapsto \mathcal{E}_1 \wedge \mathcal{E}'$ respect finite unions of subtoposes.*

Topological expressions of unions, intersections and differences of subtoposes:

Proposition. – *Let $\mathcal{E} = \widehat{\mathcal{C}}_J$ be the topos of sheaves on a site $(\mathcal{C}, J)$.*

(i) *For a family of subtoposes $\mathcal{E}_i = \widehat{\mathcal{C}}_{J_i} \hookrightarrow \widehat{\mathcal{C}}_J = \mathcal{E}$ defined by topologies J_i, $i \in I$, their union $\bigvee\limits_{i \in I} \mathcal{E}_i$ is defined by the topology $\bigwedge\limits_{i \in I} J_i$ and their intersection $\bigwedge\limits_{i \in I} \mathcal{E}_i$ is defined by the topology $\bigvee\limits_{i \in I} J_i$ generated by the topologies J_i, $i \in I$.*

(ii) *For subtoposes $\mathcal{E}_1 = \widehat{\mathcal{C}}_{J_1}$ and $\mathcal{E}_2 = \widehat{\mathcal{C}}_{J_2}$ defined by topologies J_1, J_2, their difference $\mathcal{E}_1 \backslash \mathcal{E}_2$ is defined by the topology $J_0 = (J_2 \Rightarrow J_1)$ for which a sieve C on an object X is covering if and only if*

- *for any morphism $X' \xrightarrow{x} X$ of $\mathcal{C}$, the maximal sieve is the only sieve on X' which*

$$\begin{cases} - & \text{contains } x^* C, \\ - & \text{is } J_1\text{-closed and } J_2\text{-covering.} \end{cases}$$

Reminder: A sieve C on an object X is covering for $\bigvee\limits_{i \in I} J_i$ if and only if the maximal sieve is the only sieve on X which

$$\begin{cases} - & \text{contains } C, \\ - & \text{is } J_i\text{-closed for any } i \in I. \end{cases}$$

Logical expressions of unions and intersections of subtoposes:

Proposition. –
Let $\mathcal{E} = \mathcal{E}_{\mathbb{T}}$ be the classifying topos of a geometric first-order theory $\mathbb{T}$.
Let $\mathcal{E}_i = \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}} = \mathcal{E}$, $i \in I$, be a family of subtoposes of $\mathcal{E}$ which classify quotient theories $\mathbb{T}_i$ of $\mathbb{T}$. Then:

(i) *The intersection subtopos* $\bigwedge_{i \in I} \mathcal{E}_i \longhookrightarrow \mathcal{E}$ *classifies the quotient theory* $\bigvee_{i \in I} \mathbb{T}_i$ *of* $\mathbb{T}$ *defined by the join of the families of axioms of all* $\mathbb{T}_i$*'s.*

(ii) *The union subtopos* $\bigvee_{i \in I} \mathcal{E}_i \longhookrightarrow \mathcal{E}$ *classifies any quotient theory* $\mathbb{T}'$ *of* $\mathbb{T}$ *such that a geometric sequent* $\varphi \vdash \psi$ *in the vocabulary of* $\mathbb{T}$ *is provable in* $\mathbb{T}'$ *if and only if it is provable in each* $\mathbb{T}_i$, $i \in I$.

Remark: In practice, unions $\bigvee_{i \in I} \mathcal{E}_{\mathbb{T}_i}$ can be computed if $\mathcal{E}_{\mathbb{T}}$ and its subtoposes $\mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}$ can be presented as

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J \qquad \text{and} \qquad \mathcal{E}_{\mathbb{T}_i} \cong \widehat{\mathcal{C}}_{J_i}\,,\ i \in I\,,$$

for some explicit topologies J_i, $i \in I$, on a small category $\mathcal{C}$.

Logical expressions of differences of subtoposes:

Proposition (O.C., see chapter 4 of [TST]). –
Let $\mathcal{E} = \mathcal{E}_{\mathbb{T}}$ be the classifying topos of a geometric first-order theory $\mathbb{T}$.
Let $\mathcal{E}_1 = \mathcal{E}_{\mathbb{T}_1} \hookrightarrow \mathcal{E}_{\mathbb{T}} = \mathcal{E}$ and $\mathcal{E}_2 = \mathcal{E}_{\mathbb{T}_2} \hookrightarrow \mathcal{E}_{\mathbb{T}} = \mathcal{E}$
be the classifying toposes of quotient theories $\mathbb{T}_1, \mathbb{T}_2$ of $\mathbb{T}$.
Then the difference subtopos $\quad \mathcal{E}_1 \backslash \mathcal{E}_2 \longhookrightarrow \mathcal{E}$
classifies the quotient theory of $\mathbb{T}$ $\quad \mathbb{T}' = (\mathbb{T}_2 \Rightarrow \mathbb{T}_1)$
defined from $\mathbb{T}$ by adding as axioms the geometric implications

$$\psi(\vec{y}) \vdash \psi'(\vec{y})$$

such that:

- *the reverse implication $\psi'(\vec{y}) \vdash \psi(\vec{y})$ is provable in $\mathbb{T}$,*
- *for any geometric formula $\varphi(\vec{x})$ in the vocabulary of $\mathbb{T}$,*
  for any $\mathbb{T}$-provably functional geometric formula

  $$\theta(\vec{x}, \vec{y}) : \varphi(\vec{x}) \longrightarrow \psi(\vec{y})$$

  and for any geometric formula $\chi(\vec{x})$ verifying the conditions

  $$\begin{cases} - \quad \chi(\vec{x}) \vdash \varphi(\vec{x}) \text{ is } \mathbb{T}\text{-provable,} \\ - \quad \varphi(\vec{x}) \vdash \chi(\vec{x}) \text{ is } \mathbb{T}_2\text{-provable,} \\ - \quad (\exists\, \vec{y})(\theta(\vec{x}, \vec{y}) \wedge \psi'(\vec{y})) \vdash \chi(\vec{x}) \text{ is } \mathbb{T}\text{-provable,} \end{cases}$$

  then $\quad \varphi(\vec{x}) \vdash \psi(\vec{x}) \quad$ *is $\mathbb{T}_1$-provable.*

Proof of the logical expressions of differences of subtoposes:

The proof is based on the following theorem:

Theorem. – *Let $\mathbb{T}$ be a geometric first-order theory.*

(i) *The classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ can be constructed as the topos of sheaves*

$$\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$$

on the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ endowed with the syntactic topology $J_{\mathbb{T}}$.

(ii) *The canonical functor $\ell : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}}$ is fully faithful.*

(iii) *For any object of $\mathcal{C}_{\mathbb{T}}$, i.e. any geometric formula $\varphi(\vec{x})$, the subobjects of $\ell(\varphi(\vec{x}))$ in $\mathcal{E}_{\mathbb{T}}$ correspond to subobjects of $\varphi(\vec{x})$ in $\mathcal{C}_{\mathbb{T}}$, i.e. to formulas $\chi(\vec{x})$ such that $\chi(\vec{x}) \vdash \varphi(\vec{x})$ is $\mathbb{T}$-provable.*

(iv) *In particular, any sieve on an object $\varphi(\vec{x})$ of $\mathcal{C}_{\mathbb{T}}$ has an image which is a geometric formula $\chi(\vec{x})$ such that $\chi(\vec{x}) \vdash \varphi(\vec{x})$ is $\mathbb{T}$-provable.*

Sketch of the proof of the logical expression of a difference:

Subtoposes $\mathcal{E}_{\mathbb{T}_1}$ and $\mathcal{E}_{\mathbb{T}_2}$ of $\mathcal{E}_{\mathbb{T}}$ are defined by topologies

$$J_1 \supseteq J_{\mathbb{T}} \quad \text{and} \quad J_2 \supseteq J_{\mathbb{T}} \quad \text{on} \quad \mathcal{C}_{\mathbb{T}}$$

such that, for any sieve on an object $\varphi(\vec{x})$ of $\mathcal{C}_{\mathbb{T}}$, it is covering for J_1 [resp. J_2] if and only if its image $\chi(\vec{x})$ verifies the condition that

$$\varphi(\vec{x}) \vdash \chi(\vec{x}) \text{ is provable in } \mathbb{T}_1 \text{ [resp. } \mathbb{T}_2].$$

The logical expression of existential push-forward of subtoposes:

Consider a morphism of toposes $\mathcal{E}' \xrightarrow{e} \mathcal{E}$
presented in the form $\mathcal{E}' = \widehat{\mathcal{C}}_J \longrightarrow \mathcal{E}_{\mathbb{T}} = \mathcal{E}$ which corresponds to a model M of a geometric first-order theory $\mathbb{T}$ in the topos of sheaves on a site $(\mathcal{C}, J)$.

Proposition. – *For any subtopos $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$ corresponding to a topology $J_1 \supseteq J$ and a sheafification functor $j^* : \mathcal{E}' = \widehat{\mathcal{C}}_J \longrightarrow \widehat{\mathcal{C}}_{J_1} = \mathcal{E}'_1$,*
let $\mathbb{T}_1$ be a quotient theory of $\mathbb{T}$
such that any geometric implication in the vocabulary of $\mathbb{T}$

$$\varphi(\vec{x}) \vdash \psi(\vec{x})$$

is $\mathbb{T}_1$-provable if an only if j^* transforms the embedding of $\widehat{\mathcal{C}}_J$

$$M(\varphi \wedge \psi) \hookrightarrow M\varphi$$

into an isomorphism of $\widehat{\mathcal{C}}_{J_1}$.
Then $\mathbb{T}_1$ defines the smallest subtopos

$$e_*(\mathcal{E}'_1) = \mathcal{E}_{\mathbb{T}_1} \hookrightarrow \mathcal{E}_{\mathbb{T}}$$

such that the composite morphism

$$\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \xrightarrow{e} \mathcal{E} = \mathcal{E}_{\mathbb{T}}$$

factorizes through $\quad e_*(\mathcal{E}'_1) \hookrightarrow \mathcal{E} = \mathcal{E}_{\mathbb{T}}$.

A semantic expression of pull-back of subtoposes:

We still consider a morphism of toposes $\widehat{\mathcal{C}}_J = \mathcal{E}' \xrightarrow{e} \mathcal{E} = \mathcal{E}_{\mathbb{T}}$
which corresponds to a model M of $\mathbb{T}$ in $\widehat{\mathcal{C}}_J$.

Proposition. – *For any subtopos $\mathcal{E}_1 \hookrightarrow \mathcal{E} = \mathcal{E}_{\mathbb{T}}$
corresponding to a quotient theory $\mathbb{T}_1$ of $\mathbb{T}$
defined by a list of extra axioms*

$$\varphi_i \vdash \psi_i \,, \quad i \in I \,,$$

*consider the topology J_1 on $\mathcal{C}$
which is generated by J and the stable family of sieves*

$$M(\varphi_i \wedge \psi_i) \times_{M\varphi_i} y(X)$$

associated with

- *the extra axioms $\varphi_i \vdash \psi_i$, $i \in I$,*
- *objects X of $\mathcal{C}$ embedded via $y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$,*
- *elements of $M\varphi_i(X)$ interpreted as morphisms $y(X) \to M\varphi_i$ in $\widehat{\mathcal{C}}$.*

Then the topology J_1 on $\mathcal{C}$ defines a subtopos

$$e^{-1}\mathcal{E}_1 = \widehat{\mathcal{C}}_{J_1} \longhookrightarrow \widehat{\mathcal{C}}_J = \mathcal{E}'$$

such that, for any subtopos $\mathcal{E}_1' \hookrightarrow \mathcal{E}'$,

$$e^{-1}\mathcal{E}_1 \supseteq \mathcal{E}_1' \Leftrightarrow \mathcal{E}_1 \supseteq e_* \mathcal{E}_1' \,.$$

A topological expression of push-forward of subtoposes:

Consider a morphism of toposes $\mathcal{E}' \xrightarrow{e} \mathcal{E}$
presented in the form $\mathcal{E}' \longrightarrow \mathcal{E} = \widehat{\mathcal{C}}_J$

which corresponds to a functor $\rho : \mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{e^*} \mathcal{E}'$
which is "flat" and "J-continuous".

Proposition. –
For any subtopos $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$
and the associated functor $j^* : \mathcal{E}' \to \mathcal{E}'_1$,
let $J_1 \supseteq J$ be the topology on $\mathcal{C}$
for which a sieve C on an object X of $\mathcal{C}$ is covering
if and only if its transform by

$$j^* \circ \rho : \mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{e^*} \mathcal{E}' \xrightarrow{j^*} \mathcal{E}'_1$$

is a globally epimorphic family of morphisms.

Then the subtopos defined by J_1

$$\widehat{\mathcal{C}}_{J_1} \hookrightarrow \widehat{\mathcal{C}}_J = \mathcal{E}$$

is the push-forward of $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$ by $e : \mathcal{E}' \to \mathcal{E}$

$$e_*(\mathcal{E}'_1) \hookrightarrow \mathcal{E}\,.$$

A topological expression of push-forward of subtoposes:

We still consider a morphism of toposes $\mathcal{E}' \xrightarrow{e} \mathcal{E} = \widehat{\mathcal{C}}_J$
corresponding to a functor $\rho : \mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{e^*} \mathcal{E}'$
and its unique colimit preserving extension $\widehat{\rho} : \widehat{\mathcal{C}} \to \mathcal{E}'$.
As ρ is "flat", $\widehat{\rho}$ respects finite limits.

Proposition. –
For any subtopos $\mathcal{E}_1 \hookrightarrow \mathcal{E} = \widehat{\mathcal{C}}_J$
defined by a topology $J_1 \supseteq J$ on $\mathcal{C}$,
its pull-back by the morphism $e : \mathcal{E}' \to \mathcal{E}$

$$e^{-1}\mathcal{E}_1 \longhookrightarrow \mathcal{E}'$$

is defined by the topology on $\mathcal{E}'$
generated by the monomorphisms

$$\widehat{\rho}(C) \longhookrightarrow \widehat{\rho} \circ y(X) = e^* \circ \ell(X) = \rho(X)$$

obtained as the transformations by $\widehat{\rho}$
of any family of sieves on $\mathcal{C}$

$$(C \longhookrightarrow y(X))$$

which generates the topology J_1 on $\mathcal{C}$ from J.

Correspondences and their actions on subtoposes:

Definition. –

(i) *A correspondence between a pair of toposes $\mathcal{E}$ and $\mathcal{E}'$ is a pair of topos morphisms from a third topos $\mathcal{E}_\Gamma$*

$$\mathcal{E}' \xleftarrow{p} \mathcal{E}_\Gamma \xrightarrow{q} \mathcal{E}.$$

(ii) *Such a correspondence is called "embedded" if the associated morphism*

$$\mathcal{E}_\Gamma \longrightarrow \mathcal{E}' \times \mathcal{E}$$

is an embedding.

Definition. – *The action of a correspondence $\mathcal{E}' \xleftarrow{p} \mathcal{E}_\Gamma \xrightarrow{q} \mathcal{E}$ on subtoposes is the map*

$$q_* \circ p^{-1} : \{\text{subtoposes of } \mathcal{E}'\} \longrightarrow \{\text{subtoposes of } \mathcal{E}\}.$$

Remark: Any correspondence $\mathcal{E}' \xleftarrow{p} \mathcal{E}_\Gamma \xrightarrow{q} \mathcal{E}$ defines an embedded correspondence $\mathcal{E}_{\overline{\Gamma}}$ as the image of

$$\mathcal{E}_\Gamma \longrightarrow \mathcal{E}' \times \mathcal{E}.$$

But the actions on subtoposes of $\mathcal{E}_\Gamma$ and $\mathcal{E}_{\overline{\Gamma}}$ are not the same in general, even if $p = \mathrm{id}$ and $\mathcal{E}' = \mathcal{E}'_\Gamma \xrightarrow{q} \mathcal{E}$ is a morphism.

A topological expression of the action of embedded correspondences:

Consider a pair of toposes of sheaves $\mathcal{E}' = \widehat{\mathcal{D}}_K$ and $\mathcal{E} = \widehat{\mathcal{C}}_J$.

Their product can be presented as $\mathcal{E}' \times \mathcal{E} = (\widehat{\mathcal{D} \times \mathcal{C}})_{K \times J}$
if $K \times J$ denotes the topology on $\mathcal{D} \times \mathcal{C}$ generated by K and J.

Proposition. –

Consider an embedded correspondence $\mathcal{E}_\Gamma \hookrightarrow \mathcal{E}' \times \mathcal{E} = (\widehat{\mathcal{D} \times \mathcal{C}})_{K \times J}$ corresponding to a topology Γ on $\mathcal{D} \times \mathcal{C}$ which contains K and J.
Then, for any subtopos

$$\mathcal{E}'_1 \longhookrightarrow \mathcal{E}' \quad \text{corresponding to a topology } K_1 \supseteq K \text{ on } \mathcal{D},$$

its transform by the correspondence $\mathcal{E}_\Gamma$ is the subtopos

$$\mathcal{E}_1 \longhookrightarrow \mathcal{E}$$

defined by the topology $J_1 \supseteq J$ on $\mathcal{C}$
for which a sieve

$$C \text{ on an object } X \text{ of } \mathcal{C}$$

is covering if and only if, for any object Y of $\mathcal{D}$,

$$C \text{ considered as a sieve on the object } (Y, X) \text{ of } \mathcal{D} \times \mathcal{C}$$

is covering for the topology generated by Γ and K_1.

The theorem on compatibility of pull-backs and unions of subtoposes:

For any topos morphism $e : \mathcal{E}' \to \mathcal{E}$, the associated maps

$$\{\text{subtoposes of } \mathcal{E}'\} \underset{e_*}{\overset{e^{-1}}{\rightleftarrows}} \{\text{subtoposes of } \mathcal{E}\}$$

are adjoint.
So e^{-1} respects arbitrary intersections of toposes
and e_* respects arbitrary unions.
In general, e_* does not respect even finite intersections.
On the other hand, we are going to sketch the proof of:

Theorem. – *Let $e : \mathcal{E}' \to \mathcal{E}$ be a morphism of toposes. Then:*

(i) *The induced pull-back map e^{-1} respects finite unions of toposes.*

(ii) *If the morphism e is "locally connected"*
e^{-1} even respects arbitrary unions of toposes
and, as a consequence, has a left adjoint

$$e_! : \{\text{subtoposes of } \mathcal{E}'\} \longrightarrow \{\text{subtoposes of } \mathcal{E}\}$$

characterized by the property that,
for any subtoposes $\mathcal{E}'_1 \hookrightarrow \mathcal{E}'$ and $\mathcal{E}_1 \hookrightarrow \mathcal{E}$,

$$e_!\, \mathcal{E}'_1 \supseteq \mathcal{E}_1 \iff \mathcal{E}'_1 \supseteq f^{-1}\mathcal{E}_1 \,.$$

Reminder on "locally connected" morphisms:

Definition. –

(i) *A topos morphism* $e = (e^*, e_*) : \mathcal{E}' \to \mathcal{E}$ *is called "essential" if* $e^* : \mathcal{E} \to \mathcal{E}'$ *also has a left adjoint* $e_! : \mathcal{E}' \to \mathcal{E}$.

(ii) *An essential morphism of toposes*

$$e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$$

is called "locally connected" if, for any base change morphism $\mathcal{E}_1 \xrightarrow{b} \mathcal{E}$, *the induced morphism* $f = (f^*, f_*) : \mathcal{E}'_1 = \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}_1 \to \mathcal{E}_1$ *is still essential, and the adjoint squares*

$$\begin{tikzcd} \mathcal{E}'_1 \arrow[d, "f_!"'] & \mathcal{E}' \arrow[l, "b'^*"'] \arrow[d, "e_!"] \\ \mathcal{E}_1 & \mathcal{E} \arrow[l, "b^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}'_1 \arrow[r, "b'_*"] & \mathcal{E}' \\ \mathcal{E}_1 \arrow[u, "f^*"] \arrow[r, "b_*"] & \mathcal{E} \arrow[u, "e^*"'] \end{tikzcd}$$

are commutative.

Remark: It can be proved that, in order to verify that an essential morphism $e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$ is "locally connected", it is enough to consider base changes by morphisms $\mathcal{E}_1 = \mathcal{E}/E \to \mathcal{E}$ associated to objects E of $\mathcal{E}$.

Reduction to the case of locally connected morphisms:

We already know that functors of intersections with a subtopos $\mathcal{E}' \hookrightarrow \mathcal{E}$ respect finite unions.
So part (i) of the theorem is reduced to part (ii) and the following:

Proposition. – *Any topos morphism $\mathcal{E}' \to \mathcal{E}$ factorizes as*

$$\mathcal{E}' \hookrightarrow \widehat{\mathcal{C}}'_{J'} \longrightarrow \widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E} \quad \text{where}$$

- $\mathcal{E}' \hookrightarrow \widehat{\mathcal{C}}'_{J'}$ *is an embedding of toposes,*
- $\widehat{\mathcal{C}}'_{J'} \longrightarrow \widehat{\mathcal{C}}_J$ *is induced by a fibration $\mathcal{C}' \xrightarrow{p} \mathcal{C}$,*
- $J' = p^*(J)$ *is the "Giraud topology" induced by J from $\mathcal{C}$ to $\mathcal{C}'$,*
- $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$ *is an equivalence.*

Theorem. –
If $\mathcal{C}' \xrightarrow{p} \mathcal{C}$ is a fibration
and $J' = p^*(J)$ is the "Giraud topology" on $\mathcal{C}'$
induced by a topology J on $\mathcal{C}$,
the induced topos morphism

$$p : \widehat{\mathcal{C}}'_{J'} \longrightarrow \widehat{\mathcal{C}}_J$$

is "locally connected".

Reminder on fibrations:

Definition. – *Consider a functor* $p : \mathcal{C} \to \mathcal{B}$*.*

(i) *A morphism* $x : X_1 \to X_2$ *of* $\mathcal{C}$ *is called "p-cartesian"*
if, for any morphism $x_2 : X \to X_2$ *of* $\mathcal{C}$
and any morphism $y_1 : p(X) \to p(X_1)$ *of* $\mathcal{B}$ *such that* $p(x_2) = p(x) \circ y_1$*,*
there is a unique morphism $x_1 : X \to X_1$
such that $x_2 = x \circ x_1$ *and* $p(x_1) = y_1$*.*

(ii) *The functor* $p : \mathcal{C} \to \mathcal{B}$ *is called a "fibration"*
if, for any object X_1 *of* $\mathcal{C}$ *and any morphism* $Y \xrightarrow{y_1} p(X_1)$ *of* $\mathcal{B}$*,*
there exists a p-cartesian morphism $X \xrightarrow{x_1} X_1$ *of* $\mathcal{C}$
and an isomorphism $y : p(X) \xrightarrow{\sim} Y$ *such that* $p(x_1) = y_1 \circ y$*.*

Proposition. – *Consider a fibration* $p : \mathcal{C} \to \mathcal{B}$ *of essentially small categories. Then for any functor* $\mathcal{B}' \to \mathcal{B}$ *from an essentially small category* $\mathcal{B}'$ *to* $\mathcal{B}$*, the induced fucntor* $\mathcal{C} \times_{\mathcal{B}} \mathcal{B}' \to \mathcal{B}'$ *is still a fibration, and the topos square*

$$\begin{tikzcd} \widehat{\mathcal{C} \times_{\mathcal{B}} \mathcal{B}'} \arrow[r] \arrow[d] & \widehat{\mathcal{C}} \arrow[d] \\ \widehat{\mathcal{B}'} \arrow[r] & \widehat{\mathcal{B}} \end{tikzcd}$$

is cartesian.

Reminder on Giraud topologies:

Proposition. –
Consider a fibration of essentially small categories

$$p : \mathcal{C}' \longrightarrow \mathcal{C}$$

and the essential morphism it defines

$$p = (p_!, p^*, p_*) : \widehat{\mathcal{C}'} \longrightarrow \widehat{\mathcal{C}}.$$

Then, for any subtopos

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}},$$

its pull-back by p

$$\widehat{\mathcal{C}'}_{J'} \hookrightarrow \widehat{\mathcal{C}'}$$

is the subtopos defined by the "Giraud topology" J' on $\mathcal{C}'$ for which

$$\begin{cases} \text{a sieve } C \text{ on an object } X \text{ of } \mathcal{C}' \text{ is covering} \\ \text{if and only if the images by } p \\ \qquad p(x) : p(X') \longrightarrow p(X) \\ \text{of the } p\text{-cartesian morphisms contained in } C \\ \qquad x : X' \longrightarrow X \\ \text{make up a } J\text{-covering family of the object } p(X) \text{ of } \mathcal{C}. \end{cases}$$

Corollary. – *Consider a fibration $p : \mathcal{C}' \to \mathcal{C}$ of essentially small categories. Then:*

(i) *The map*

$$\begin{array}{rcl} \{\textit{topologies on } \mathcal{C}\} & \longrightarrow & \{\textit{topologies on } \mathcal{C}'\} \\ J & \longmapsto & J' = \textit{ Giraud topology induced by } J \end{array}$$

respects arbitrary intersections of topologies. In other words, the map

$$p^{-1} : \{\textit{subtoposes of } \widehat{\mathcal{C}}\} \longrightarrow \{\textit{subtoposes of } \mathcal{C}'\}$$

respects arbitrary unions of subtoposes.

(ii) *For any topology J on $\mathcal{C}$ and the induced Giraud topology J' on $\mathcal{C}'$, the functor of composition with p*

$$p^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}'$$

transforms J-sheaves into J'-sheaves and respects arbitrary limits. In other words, there are two adjoint commutative squares:

$$\begin{tikzcd} \widehat{\mathcal{C}}'_{J'} \arrow[d, dash, "p_!"'] & \widehat{\mathcal{C}}' \arrow[l, "j'^*"'] \arrow[d, dash, "p_!"] \\ \widehat{\mathcal{C}}_J & \widehat{\mathcal{C}} \arrow[l, "j^*"'] \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{C}}'_{J'} \arrow[r, hook, "j'_*"] & \widehat{\mathcal{C}}' \\ \widehat{\mathcal{C}}_J \arrow[u, "p^*"] \arrow[r, hook, "j_*"] & \widehat{\mathcal{C}} \arrow[u, "p^*"'] \end{tikzcd}$$

Factorization of topos morphisms:

- Consider an arbitrary topos morphism $\mathcal{E}' \xrightarrow{e} \mathcal{E}$.
- We can write $\quad \mathcal{E} \cong \widehat{\mathcal{C}}_J, \quad \mathcal{E}' \cong \widehat{\mathcal{D}}_K$
  where $\mathcal{C} \hookrightarrow \mathcal{E}$, $\mathcal{D} \hookrightarrow \mathcal{E}'$ are small full subcategories
  such that $e^* : \mathcal{E} \to \mathcal{E}'$ restricts to a functor $\rho : \mathcal{C} \to \mathcal{D}$.

Theorem (O.C., see [Denseness]). – *Consider the small category $\mathcal{C}' = \mathcal{D}/\mathcal{C}$ whose*

- *objects are triplets $(Y, X, Y \to \rho(X))$ consisting in objects Y of $\mathcal{D}$, X of $\mathcal{C}$ and a morphism $Y \xrightarrow{t} \rho(X)$ of $\mathcal{D}$,*
- *morphisms $(Y_1, X_1, Y_1 \xrightarrow{t_1} \rho(X_1)) \longrightarrow (Y_2, X_2, Y_2 \xrightarrow{t_2} \rho(X_2))$ are pairs of compatible morphisms $(Y_1 \xrightarrow{y} Y_2, X_1 \xrightarrow{x} X_2)$.*

Let K' and J' be the topologies on $\mathcal{C}' = \mathcal{D}/\mathcal{C}$ induced by K and J via the forgetful functors $\mathcal{D}/\mathcal{C} \to \mathcal{D}$ and $\mathcal{D}/\mathcal{C} \to \mathcal{C}$. Then:

(i) *The morphism $\widehat{\mathcal{C}}'_{K'} \to \widehat{\mathcal{D}}_K$ induced by $\mathcal{C}' = \mathcal{D}/\mathcal{C} \to \mathcal{D}$ is an equivalence of toposes.*

(ii) *The topology K' contains J', and there is an embedding $\widehat{\mathcal{C}}'_{K'} \hookrightarrow \widehat{\mathcal{C}}'_{J'}$.*

(iii) *The forgetful functor $\mathcal{C}' = \mathcal{D}/\mathcal{C} \to \mathcal{C}$ is a fibration and J' is the "Giraud topology" induced by J.*

(iv) *The topos morphism $\mathcal{E}' \xrightarrow{e} \mathcal{E}$ factorizes as $\quad \widehat{\mathcal{D}}_K \cong \widehat{\mathcal{C}}'_{K'} \hookrightarrow \widehat{\mathcal{C}}'_{J'} \longrightarrow \widehat{\mathcal{C}}_J$.*

Galois correspondences between subobjects:

Lemma. –
Consider an essential morphism of toposes $e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$.
For any object E' *of* $\mathcal{E}'$*, consider the two maps*

$$\{\text{subobjects } C' \hookrightarrow E'\} \underset{G_{E'}}{\overset{F_{E'}}{\rightleftarrows}} \{\text{subobjects } C \hookrightarrow e_! E'\}$$

defined by

$$F_{E'}(C' \hookrightarrow E') = (\mathrm{Im}\, e_! C' \hookrightarrow e_! E'),$$
$$G_{E'}(C \hookrightarrow e_! E') = (e^* C \times_{e^* e_! E'} E' \hookrightarrow E').$$

Then, these maps respect the order relations $\subseteq$ *on these sets, and* $F_{E'}$ *is left adjoint of* $G_{E'}$.

Corollary. –

(i) *There is an induced one-to-one correspondence between the* $(C' \hookrightarrow E')$ *which are fixed under* $G_{E'} \circ F_{E'}$ *and the* $(C \hookrightarrow e_! E')$ *which are fixed under* $F_{E'} \circ G_{E'}$.

(ii) *For any* $C' \hookrightarrow E'$*, its image under* $G_{E'} \circ F_{E'}$ *is the smallest fixed point which contains it.*

(iii) *For any* $C \hookrightarrow e_! E'$*, its image under* $F_{E'} \circ G_{E'}$ *is the biggest fixed point which is contained in it.*

Union of fixed points:

We still consider an essential morphism $e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$.

Lemma. –
For any family of subobjects of an object E' of $\mathcal{E}'$

$$C'_k \hookrightarrow E', \ k \in K, \text{ which are fixed under } G_{E'} \circ F_{E'},$$

their union

$$\bigvee_{k \in K} C'_k \hookrightarrow E' \text{ is fixed under } G_{E'} \circ F_{E'}.$$

Proof:
If each $C'_k \hookrightarrow E'$ corresponds to a fixed subobject $C_k \hookrightarrow e_! E'$, the formulas

$$C'_k = e^* C_k \times_{e^* e_! E'} E', \ \ k \in K,$$

induce the formula

$$\bigvee_{k \in K} C'_k = e^* \left(\bigvee_{k \in K} C_k \right) \times_{e^* e_! E'} E'.$$

Corollary. –
For any object E' of C',
any subobject $C' \hookrightarrow E'$
contains a biggest fixed subobject

$$\overset{\circ}{C}{}' \hookrightarrow C' \hookrightarrow E'.$$

Stability of fixed points:

Lemma. –
*Consider an essential morphism of toposes $e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$.
Then:*

(i) *For any morphism $E_2' \to E_1'$ of $\mathcal{E}'$, the map*

$$(C' \hookrightarrow E_1') \longmapsto (C' \times_{E_1'} E_2' \hookrightarrow E_2')$$

transforms

any image under $G_{E_1'}$ of some $C \hookrightarrow e_! E_1'$
into the image under $G_{E_2'}$ of $C \times_{e_! E_1'} e_! E_2'$.

(ii) *For any object E of $\mathcal{E}$ and any subobject $C \hookrightarrow E$,*
the associated subobject

$$e^* C \hookrightarrow e^* E$$

is the image under $G_{e^* E}$ of the subobject

$$C \times_E e_! e^* E \hookrightarrow e_! e^* E.$$

Proof:
(i) comes from the fact that e^* respects fiber products.
(ii) Indeed, if $C_1 = C \times_E e_! e^* E \hookrightarrow e_! e^* E$, we have

$$e^* C_1 \times_{e^* e_! e^* E} e^* E = (e^* C \times_{e^* E} e^* e_! e^* E) \times_{e^* e_! e^* E} e^* E = e^* C \hookrightarrow e^* E.$$

Characterization of pull-backs under essential morphisms:

Theorem (O.C., L.L., to appear in [Engendrement]). –
Consider an essential morphism of toposes $e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}$. Then for any subtoposes defined by a topology J on $\mathcal{E}$

$$\mathcal{E}_J \longhookrightarrow \mathcal{E}\,,$$

its pull-back under $e : \mathcal{E}' \to \mathcal{E}$ is defined by the topology J' consisting in monomorphisms $C' \hookrightarrow E'$ of $\mathcal{E}'$ verifying the condition that

$$\begin{cases} \text{there exists a monomorphism } (C \hookrightarrow e_! E') \text{ in } J \\ \text{such that } (C' \hookrightarrow E') \text{ contains } (e^* C \times_{e^* e_! E'} E' \hookrightarrow E'). \end{cases}$$

Proof:

- The topology J' on $\mathcal{E}'$ which defines the pull-back of $\mathcal{E}_J \hookrightarrow \mathcal{E}$ is generated by the monomorphisms $(e^* C \hookrightarrow e^* E)$ induced by elements $(C \hookrightarrow E)$ of J.
- According to part (ii) of the previous lemma, all these generators belong to the class J'' of monomorphisms $(C' \hookrightarrow E')$ which verify the above condition.
- As J' is stable, we also have that $J'' \subseteq J'$.
- To conclude, we need to prove that J'' is a topology on $\mathcal{E}'$.

Verification of the topology axioms:

- We are reduced to proving that the class J'' of monomorphisms $(C' \hookrightarrow E')$ such that there exists $(C \hookrightarrow e_! E')$ in J verifying $\qquad e^* C \times_{e^* e_! E'} E' \subseteq C' \qquad$ is a topology on $\mathcal{E}'$.
- It obviously verifies the maximality axioms.
- Stability results from part (i) of the previous lemma.
- For transitivity, consider two monomorphisms of $\mathcal{E}'$

$$C' \hookrightarrow E' \quad \text{and} \quad D' \hookrightarrow E'$$

and a globally epimorphic family $(E'_k \to D')_{k \in K}$
such that there exist elements of J

$$D \hookrightarrow e_! E' \quad \text{and} \quad C_k \hookrightarrow e_! E'_k, \ k \in K,$$

verifying

$$D' \supseteq e^* D \times_{e^* e_! E'} E' \quad \text{and} \quad C' \times_{E'} E'_k \supseteq e^* C_k \times_{e^* e_! E'_k} E'_k, \ k \in K.$$

The family of morphisms $e_! E'_k \to e_! D'$, $k \in K$, is still globally epimorphic.
Let $C \hookrightarrow e_! E'$ be the union of the images of the morphisms

$$C_k \longrightarrow e_! E'_k \longrightarrow e_! D' \longrightarrow e_! E'.$$

Then

$$\begin{cases} - \quad \text{the monomorphism } C \hookrightarrow e_! E' \text{ belongs to } J, \\ - \quad \text{the subobject } C' \hookrightarrow E' \text{ contains } e^* C \times_{e^* e_! E'} E', \end{cases}$$

which proves that J'' verifies transitivity.

Characterization of pull-backs under locally connected morphisms:

Theorem (O.C., L.L., to appear in [Engendrement]). –
Consider a locally connected topos morphism

$$e = (e_!, e^*, e_*) : \mathcal{E}' \to \mathcal{E}.$$

Consider a subtopos defined by a topology J *on* $\mathcal{E}$

$$\mathcal{E}_J \hookrightarrow \mathcal{E}$$

and its pull-back by e *defined by a topology* J' *on* $\mathcal{E}'$

$$\mathcal{E}'_{J'} \hookrightarrow \mathcal{E}'.$$

Then a monomorphism of $\mathcal{E}'$

$$C' \hookrightarrow E'$$

belongs to J' *if and only if its biggest fixed subobject*

$$\mathring{C}' \hookrightarrow E'$$

corresponds to a fixed subobject

$$C \hookrightarrow e_! E'$$

which belongs to J.

As this characterization respects intersections of topologies, we get:

Corollary. – *If a topos morphism* $e : \mathcal{E}' \to \mathcal{E}$ *is locally connected, the associated pull-back map* e^{-1} *on subtoposes respects arbitrary unions, so has a left adjoint* $e_!$.

Characterization in the case of fixed points:

We consider the locally connected morphism

$$e = (e_!, e^*, e_*) : \mathcal{E}' \longrightarrow \mathcal{E},$$

a subtopos $\mathcal{E}_J \hookrightarrow \mathcal{E}$ defined by a topology J
and its pull-back $\mathcal{E}'_{J'} \hookrightarrow \mathcal{E}'$ defined by a topology J'.
The proof of the theorem reduces to:

Lemma. – *For any object E' of $\mathcal{E}'$ and any fixed subobject $C' \hookrightarrow E'$, which corresponds to a fixed subobject $C \hookrightarrow e_! E'$, the monomorphism $C' \hookrightarrow E'$ belongs to J' if and only if $C \hookrightarrow e_! E'$ belongs to J.*

Proof:

- As $C' = e^* C \times_{e^* e_! E'} E'$, $C' \hookrightarrow E'$ belongs to J' if $C \hookrightarrow e_! E'$ belongs to J.
- The implication in the reverse direction
  is a consequence of the commutativity of the square

$$\begin{tikzcd}
\mathcal{E}'_{J'} \arrow[d, "e_!"'] & \mathcal{E}' \arrow[l, "j'^*"'] \arrow[d, "e_!"] \\
\mathcal{E}_J & \mathcal{E} \arrow[l, "j^*"']
\end{tikzcd}$$

which is part of the definition of "local connectedness" of e.