From quantales to a Grothendieck monoidal topology: Towards a closed monoidal generalization of topos

Hugo Luiz Mariano
University of São Paulo

Toposes in Mondovì
Istituto Grothendieck
September 9-11, 2024

Abstract

In this talk, we will present some recent developments associated with some PhD theses in IME-USP (Institute of Mathematics and Statistics, University of São Paulo, Brazil) on categories of sheaves over quantales and categories of quantale valued sets, returning to a theme of studies involving logic and categories carried out at IME-USP in the latter half of the 1990, but now from a new perspective: considering semicartesian and commutative quantales, as non-idempotent generalizations of locales (= complete Heyting algebras).

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We will list some properties of the (monoidal) categories obtained, indicating some similarities and differences with the Grothendieck topos. The main goal of these efforts is to develop a closed monoidal but not cartesian closed generalization of the notion of elementary topos, in order to cover some mathematical situations (including generalizations of metric spaces), to enable an axiomatic study of these categories, and a general definition of their internal logic, which shows clues of being some form of linear logic.

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A future goal is to establish a precise relationship between the present approach and the enriched category approach to sheaves over quantales (and quantaloids) developed by I. Stubbe.

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Outline

Part (I): Preliminaries

Part (II): Towards a monoidal topos theory

Part (I): Preliminaries

(I.1) Presheaves and sheaves

- A **presheaf** (of sets) on a small category $\mathcal{C}$ is a (contra)variant functor $F : \mathcal{C}^{op} \rightarrow Set$
- A morphism of presheaves is just a natural transformation $\eta : F \rightarrow F'$
- $pSh(\mathcal{C}) := Set^{\mathcal{C}^{op}}$, the category of all presheaves and natural transformations

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- If X is a topological space, then the family of continuous real functions $(Cont(U, \mathbb{R}))_{U \in Open(X)}$ determines, under restriction, a **sheaf** C of sets (or rings) since:

  (*) for any $U \in Open(X)$ and any compatible family of continuous functions defined in a covering of U, $\bigcup_{i \in I} U_i = U$, admits a unique continuous gluing defined on U

(I.1) Presheaves and sheaves

$$C(U) \xrightarrow{\text{equalizer}} \prod_{i\in I} C(U_i) \overset{proj_l}{\underset{proj_r}{\rightrightarrows}} \prod_{i,j\in I} C(U_i \cap U_j)$$

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- The notion of sheaf over X depends only on the poset $(Open(X), \subseteq)$: this is a locale.
- $(L, \leq)$ is a **locale** if it is a complete lattice such that $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i$
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- Let $(L, \leq)$ be a locale, $F : (L, \leq)^{op} \to Set$ be a presheaf. Then F is a sheaf if for any $u \in L$ and any covering $\bigvee_{i\in I} u_i = u$, we have an equalizer

$$F(u) \xrightarrow{\text{equalizer}} \prod_{i\in I} F(u_i) \underset{proj_r}{\overset{proj_l}{\rightrightarrows}} \prod_{i,j\in I} F(u_i \wedge u_j)$$

(I.2) Localic Topos and Grothendieck Topos

- $Sh(L) \hookrightarrow pSh(L)$ is the full subcategory of all sheaves over the locale L
- In $Sh(L)$, $Sub(Hom(-, u)) \cong [0, u] \subseteq L$
- A **localic topos**, $\mathcal{E}$, is a category that is equivalent to $Sh(L)$, for some locale L.

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- A **Grothendieck pretopology** on a small category $\mathcal{C}$ *with finite limits (or just pullbacks)* is a certain family $Cov = (Cov(u))_{u \in Obj(\mathcal{C})}$, where each member of $Cov(u)$ is a family of $\mathcal{C}$-arrows with codomain u.

A **Grothendieck pretopology** on $\mathcal{C}$ associates to each object U of $\mathcal{C}$ a set $K(U)$ of families of morphisms $\{U_i \to U\}_{i \in I}$ satisfying some rules:

1. The singleton family $\{U' \xrightarrow{f} U\}$, formed by an isomorphism $f : U' \overset{\cong}{\to} U$, is in $K(U)$;
2. If $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is in $K(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j \in J_i}$ is in $K(U_i)$ for all $i \in I$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i \in I, j \in J_i}$ is in $K(U)$;
3. If $\{U_i \to U\}_{i \in I}$ is in $K(U)$, and $V \to U$ is any morphism in $\mathcal{C}$, then the family of pullbacks $\{V \times_U U_i \to V\}$ is in $K(V)$.

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For locales:
$(u_i \leq u)_{i \in I} \in Cov(u)$ iff $\bigvee_{i \in I} u_i = u$;
(Ax.3) $v \leq u \Rightarrow \bigvee_{i \in I}(u_i \wedge v) = (\bigvee_{i \in I} u_i) \wedge v = u \wedge v = v$

- $F : \mathcal{C}^{op} \to Set$ is a **sheaf** w.r.t. the pretopology Cov iff for any $u \in Obj(\mathcal{C})$ and any covering $\{f_i : u_i \to u\}_{i \in I} \in Cov(u)$, we have an equalizer

$$F(u) \xrightarrow{(F(f_i))_i} \prod_{i \in I} F(u_i) \underset{proj_r}{\overset{proj_l}{\rightrightarrows}} \prod_{i,j \in I} F(u_i \times_u u_j)$$

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- Set, $pSh(\mathcal{C})$, $Sh(L)$ are Grothendieck toposes.

- A category $\mathcal{E}$ is a Grothendieck topos iff there is a small category $\mathcal{D}$ and a pair of functors $\mathcal{E} \underset{a}{\overset{j}{\rightleftarrows}} pSh(\mathcal{D})$ such that:

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- A category is a Grothendieck topos iff:
  (i) is locally small;
  (ii) is complete and cocomplete;
  (iii) is κ-accessible, for some regular cardinal κ;
  (iv) admits exponentiation (it is cartesian closed);
  (v) has a subobject classifier

(I.3) Elementary Topos

- An **elementary topos** is a category $\mathcal{E}$ such that:
  (i) is locally small;
  (ii) has finite limits (and finite colimits);
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$$\begin{array}{ccc} a & \xrightarrow{\quad m \quad} & b \\ {\scriptstyle !_a}\downarrow & \text{p.b.} & \downarrow{\scriptstyle \chi_m} \\ 1 & \xrightarrow{\quad \top \quad} & \Omega \end{array}$$

- $\forall a \in Obj(\mathcal{E})$, $[m] \in Sub(a) \overset{\cong}{\mapsto} \chi_m \in \mathcal{E}(a, \Omega)$ is a natural bijection

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- $\forall a \in Obj(\mathcal{E})$, $[m] \in Sub(a) \overset{\cong}{\mapsto} \chi_m \in \mathcal{E}(a, \Omega)$ is a natural bijection
- Every Grothendieck topos is an elementary topos
- Set_{fin} is an elementary topos that is not a Grothendieck topos

(I.4) Internal logic of a topos

- In $Sh(L)$, $\Omega : L^{op} \to Set$, $\Omega(u) = [0, u]$,
  $v \leq u \mapsto (u' \in [0, u] \mapsto u' \wedge v \in [0, v])$
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- $\Omega \in Obj(\mathcal{E})$ encodes an intuitionist (propositional) logic internal to $\mathcal{E}$
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- Ω is an internal Heyting algebra $\frac{a \wedge b \leq c}{a \leq b \to c}$
- The internal operation $\wedge : \Omega \times \Omega \to \Omega$ corresponds to external (natural) operations $(\wedge_a : Sub(a) \times Sub(a) \to Sub(a))_{a \in Obj(\mathcal{E})}$

$$\begin{array}{ccc}
\mathcal{E}(a, \Omega \times \Omega) & \xrightarrow{\wedge \circ -} & \mathcal{E}(a, \Omega) \\
\cong \downarrow & & \downarrow = \\
\mathcal{E}(a, \Omega) \times \mathcal{E}(a, \Omega) & \longrightarrow & \mathcal{E}(a, \Omega) \\
\cong \downarrow & & \downarrow \cong \\
Sub(a) \times Sub(a) & \xrightarrow{\wedge_a} & Sub(a)
\end{array}$$

(I.5) Localic topos: alternative descriptions

- A L-**set** is a pair (X, δ) where $\delta : X \times X \to L$ is a L-fuzzy partial equivalence relation on X: (Sym) $\delta(x, y) = \delta(y, x)$; (Trans) $\delta(x, y) \wedge \delta(y, z) \leq \delta(x, z)$
- It holds: (Ide) $\delta(x, x) \wedge \delta(x, y) \wedge \delta(y, y) = \delta(x, y)$
- Let $(X, \delta), (X', \delta')$ be L-sets:
  functional morphism: $f : X \to X'$ such that...;
  relational morphism: $r : X \times X' \to L$ such that...
- Various special subclasses of L-sets: in particular, the (Scott-)complete L-sets
- $L\text{-}sets^c_{func} \underset{comp}{\overset{incl}{\rightleftarrows}} L\text{-}sets_{func}$, $incl \dashv comp$
- $Sh(L) \simeq L\text{-}sets^c_{func} \simeq L\text{-}sets^c_{rel} \cong L\text{-}sets_{rel} \simeq V^L / \equiv$

Part (II): Towards a monoidal topos theory

(II.1) Motivations and previous works

Motivations:

- Mathematical practice: new notions and examples of sheaves
- Categories with a non-intuitionistic internal logic

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Previous works (middle 1990):

- Miraglia-Coniglio: Q-sets, Q a right sided idempotent quantale
- Ulrich Höhle: Weak topos (fuzzy sets)

(II.2) Quantales: a generalization of locales

- A **quantale** is a structure $(Q, \otimes, \leq)$ such that: (i) $(Q, \leq)$ is a complete lattice; (ii) $(Q, \otimes)$ is a semigroup; (iii) distributive laws: $a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \otimes b_i$; $(\bigvee_{i \in I} b_i) \otimes a = \bigvee_{i \in I}(b_i \otimes a)$
- **Examples:** (a) $(L, \wedge, \leq)$ locale; (b) $(Ideals(A), \cdot, \subseteq)$, A a commutative ring with 1; (c) $([0, \infty], +, \geq)$ Lawvere quantale; (d) $(clrgId(B), \cdot, \subseteq)$, B a C^*-algebra

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- **Some subclasses:** (i) Q is **unital** whenever $(Q, \otimes)$ is a monoid; (ii) Q is **idempotent** whenever $a \otimes a = a, \forall a$; (iii) Q is **semicartesian** whenever $a \otimes b \leq a \wedge b$
- unital + semicartesian = $\top$ is the identity; idempotent+ semicartesian = locale
- Examples (a), (b), (c) are commutative, unital and semicartesian; (a), (b) are non-idempotent (in general); (d) is idempotent, right-sided and non-commutative (in general)

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  Example: $(x) \in Ideals(A) \mapsto A[x^{-1}]$

- A Q-**set** is a pair (X, δ) where $\delta : X \times X \to Q$ is a Q-fuzzy partial equivalence relation on X (or a Q-valued pseudometric on X):
  (Sym) $\delta(x, y) = \delta(y, x)$;
  (Trans) $\delta(x, y) \otimes \delta(y, z) \leq \delta(x, z)$;
  (Ide) $\delta(x, x) \otimes \delta(x, y) = \delta(x, y) = \delta(x, y) \otimes \delta(y, y)$
  Note that $Ex := \delta(x, x) \in Idem(Q)$

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Q-valued sets: José Alvim, Caio Mendes

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  (v) is a monoidal semicartesian category $X \otimes X' = \{(x, x') \in X \times X' : Ex = E'x'\}$; $\delta_\otimes((x, x'), (y, y')) := \delta(x, y) \otimes \delta'(x', y')$;
  (vi) admits exponentiation: $Q\text{-}sets_{func}(a \otimes b, c) \cong Q\text{-}sets_{func}(a, c^b)$;

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  (vi) admits exponentiation: $Q\text{-}sets_{func}(a \otimes b, c) \cong Q\text{-}sets_{func}(a, c^b)$;
  (vii) has a classifier for regular subobjects $1 \rightarrowtail \Omega$ $\Omega = Idem(Q) \dot\sqcup Idem(Q)$, $\delta((e, i), (e', i')) = e \otimes e'$

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- (X, δ) is a (Scott)-complete Q-set iff:
  "every singleton on X ($\sigma : X \to Q$ such that...) is uniquely represented ($\exists ! x$, $\sigma = \delta(x, -)$)"
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- Q is **strong quantale** iff $\forall e \in Idem(Q)$,
$e \leq \bigvee_i a_i \;\Rightarrow\; e \leq \bigvee_i a_i \otimes a_i$
- If Q is commutative, semicartesian and strong quantale:
$Q\text{-}sets^{c}_{func} \;\simeq\; Q\text{-}sets^{c}_{rel} \;\cong\; Q\text{-}sets_{rel}$

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  (ix) Quantales with extra properties:
  - best idempotent approximations: $u^- \leq u \leq u^+$;
  - variant notions of suboject classifiers in $Sh(Q)$:
  $\Omega^-(u) = \{q \in Q : q \otimes u^- = q\}$,
  $v \leq u \mapsto (q \in \Omega^-(u) \mapsto q \otimes v^- \in \Omega^-(v))$
  $\Omega^+(u) = \{q \in Q : q^+ \otimes u = q\}$,
  $v \leq u \mapsto (q \in \Omega^+(u) \mapsto q^+ \otimes v \in \Omega^+(v))$

(II.6) Generalization: Lopologies

Grothendieck lopologies (examples, results, applications): Ana Luiza Tenório

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A **Grothendieck prelopology** on a *small monoidal semicartesian category* $(\mathcal{C}, \otimes, 1)$ *with finite limits (or just terminal 1 and equalizers)* is a certain family $Cov = (Cov(u))_{u \in Obj(\mathcal{C})}$, where each member of $Cov(u)$ is a family of $\mathcal{C}$-arrows with codomain u.

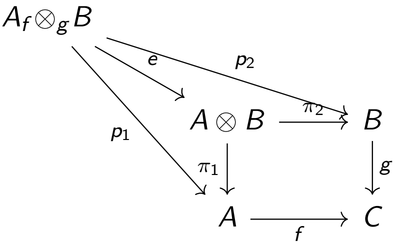
The pseudo-pullback

Let $(\mathcal{C}, \otimes, 1)$ be a semicartesian monoidal category with equalizers.

(II.6) Generalization: Lopologies

The pseudo-pullback

Let $(\mathcal{C}, \otimes, 1)$ be a semicartesian monoidal category with equalizers. The **pseudo-pullback** of morphism $f : A \to C$ and $g : B \to C$ is the equalizer of the parallel arrows $A \otimes B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$ where $\pi_1 = \rho_A \circ (id_A \otimes !_B)$ and $\pi_2 = \lambda_B \circ (!_A \otimes id_B)$.



Grothendieck prelopology

Let $(\mathcal{C}, \otimes, 1)$ be a semicartesian monoidal category with equalizers. A **Grothendieck prelopology** on $\mathcal{C}$ associates to each object U of $\mathcal{C}$ a set $L(U)$ of families of morphisms $\{U_i \to U\}_{i \in I}$ such that:

(1) The singleton family $\{U' \xrightarrow{f} U\}$, formed by an isomorphism $f : U' \overset{\cong}{\to} U$, is in $L(U)$;

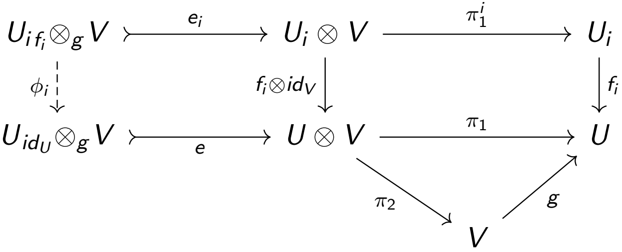
(2) If $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is in $L(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j \in J_i}$ is in $L(U_i)$ for all $i \in I$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i \in I, j \in J_i}$ is in $L(U)$;

(3) If $\{f_i : U_i \to U\}_{i \in I} \in L(U)$, then $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i \in I}$ is in $L(U \otimes V)$ and $\{id_V \otimes f_i : V \otimes U_i \to V \otimes U\}_{i \in I}$ is in $L(V \otimes U)$, for any V object in $\mathcal{C}$

(II.6) Generalization: Lopologies

(4) If $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is in $L(U)$ and $g : V \to U$ is any morphism in $\mathcal{C}$, then $\{\phi_i : U_{i\,f_i}\otimes_g V \to U_{id_U}\otimes_g V\}_{i\in I}$ is in $L(U_{id_U}\otimes_g V)$ and $\{\phi_i : V_g\otimes_{f_i} U \to V_g\otimes_{id_U} U\}_{i\in I}$ is in $L(V_g\otimes_{id_U} U)$.

For each $i \in I$, the arrow $\phi_i : U_{i\,f_i}\otimes_g V \to U_{id_U}\otimes_g V$ is unique because of the universal property of the equalizer:



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Proposition

Let $(\mathcal{C}, \times, 1)$ is a cartesian category with equalizers, then are equivalent:

- L is a Grothendieck **prelopology** in $\mathcal{C}$;
- L is a Grothendieck **pretopology** in $\mathcal{C}$.

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Fact

A product of prelopogies is a prelopology in the product category.

(II.7) Grothendieck Lopos

- $F : \mathcal{C}^{op} \to Set$ is a sheaf w.r.t. the prelopology L iff for any $u \in Obj(\mathcal{C})$ and any covering $\{f_i : u_i \to u\}_{i \in I} \in L(u)$, we have an equalizer

$$F(u) \xrightarrow{(F(f_i))_i} \prod_{i \in I} F(u_i) \underset{proj_r}{\overset{proj_l}{\rightrightarrows}} \prod_{i,j \in I} F(u_i \otimes_u u_j)$$

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- A category $\mathcal{E}$ is a **Grothendieck lopos** iff $\mathcal{E} \simeq Sh((\mathcal{C}, \otimes, 1), L)$, for some small semicartesian category $(\mathcal{C}, \otimes, 1)$ endowed with a Grothendieck prelopology.

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Remark: A sufficient condition that guarantees that $Sh(\mathcal{C}, L)$ is closed monoidal category is: for all U, $F(U \otimes -)$ is a sheaf whenever F is. This condition is satisfied whenever:
(a) $\mathcal{C}$ is a quantale, or
(b) $U \otimes -$ preserves equalizers, or
(c) K is a strong lopology in $\mathcal{C}$ (i.e., holds another technical axiom)

Another possible approach to Grothendieck lopos:

A category $\mathcal{E}$ is a **Grothendieck lopos** if and only if $\mathcal{E}$ has a small set of generators and $Y : \mathcal{E} \to Set^{\mathcal{E}^{op}}$ has a left adjoint monoidal functor that preserves pseudo-pullback.

(II.8) Future works

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- Establish relationships with the work of Isar Stubbe on enriched categories over quantaloids

Main references:

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Thank you very much!!!