Morphisms and comorphisms of sites: double-categorical and profunctorial aspects

Axel Osmond, *joint work with* Olivia Caramello

ISTITUTO GROTHENDIECK

TOPOSES IN MONDOVÌ

Introduction

In practice, topoi are often presented through sites.

As well, geometric morphisms are often presented by functors between those sites.

Two dual classes of functors between sites induce geometric morphisms:

- *morphisms of sites*, characterized by some preservation property
- *comorphisms of sites*, characterized by some reflection property

In this talk we will try to understand the reason of this dichotomy, and how to reconciliate those to classes of functors:

- either by arranging them altogether in a suited 2-dimensional structure
- either by exhibiting them as two instances of a same notion

Outlay

Morphisms and comorphisms of sites

Sieves

Grothendieck topologies and coverages can be expressed in several ways, either through *covering families*, or also through *sieves*.

Definition

A *sieve* on an object c in a category $\mathcal{C}$ is a subobject of the representable $S \rightarrowtail \text{よ}_c$.

Equivalently, a family S of arrows with codomain c absorbing under precomposition: if $u : c' \to c$ is in S, then for any $u' : c'' \to c'$ the composite $uu' : c'' \to c$ is in S.

For a family of arrows S with common codomain c, the sieve *generated* by S is the set of maps that factorizes through a map in S

$$\overline{S} = \left\{ v : d \to c \mid \exists u : c' \to c \in S \text{ such that } \begin{array}{ccc} d & \xrightarrow{v} & c \\ {\scriptstyle \exists}\downarrow & \nearrow_{u} & \\ c' & & \end{array} \right\}$$

For two sieves S, R on c, $R \leq S$ if any arrow in R factorizes through an arrow in S.

Coverage and sites

Definition

A *Grothendieck coverage* J on $\mathcal{C}$ consists for each object c of a set $J(c)$ of sieves on c that are declared *J-covering*, such that

- for each c, the *maximal sieve*, which is $よ_c$, is in $J(c)$
- for each arrow $a : d \to c$ and each S in $J(c)$, the *pullback sieve* below is in $J(d)$

$$a^*S = \left\{ v : d' \to d \mid \exists u : c' \to c \in S \text{ and a factorization } \begin{array}{ccc} d' & \overset{\exists}{\dashrightarrow} & c' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u \in S} \\ d & \underset{a}{\longrightarrow} & c \end{array} \right\}$$

This can be completed with additional axioms, as the axiom of *locality*, in order to define *Grothendieck topologies*; however in this talk we will only consider coverages.

Definition

A *site* is a pair $(\mathcal{C}, J)$ with J a coverage on $\mathcal{C}$.

Sites in this talk will be considered as *small generated*, that is, generated from a small category, even when they are large.

Notions of morphisms between sites

A functor between sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ may behave in two relevant ways relative to the coverages:

- either by *preserving* covering sieves
- either by *reflecting* covering sieves

Combined with flatness conditions (which we will not discuss much in this talk), cover preservation allows to define a notion of *morphism of sites*.

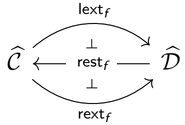
On the other hand cover reflection allows to define a notion of *comorphism of sites*

Both induce geometric morphisms between associated sheaf topoi.

Let us revisit those ideas through the formalism of *extensions and restrictions*.

Extension and restriction

Recall that any functor $f : \mathcal{C} \to \mathcal{D}$ induces a triple of adjoints



where lext_f (resp. rext_f) sends a presheaf $X : \mathcal{C}^{\text{op}} \to \textbf{Set}$ to its left (resp. right) Kan extension along f^{op}

$$\text{lext}_f X = \text{lan}_{f^{\text{op}}} X \qquad\qquad \text{rext}_f X = \text{ran}_{f^{\text{op}}} X$$

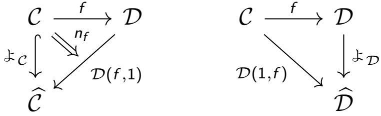
while the restriction functor rest_f sends a presheaf $Y : \mathcal{D}^{\text{op}} \to \textbf{Set}$ to the precomposite

$$\text{rest}_f Y = Y(f(-))$$

Beware that lext and rest are covariant in f, while rest is contravariant in f.

Extension and restriction from the nerves

Extensions and restrictions functors can also be constructed formally from the left and right nerves depicted below:



Proposition

For any functor $f : \mathcal{C} \to \mathcal{D}$ the nerve satisfies the following identities

$$\begin{aligned} \mathrm{rest}_f &= \mathrm{lan}_{\text{よ}_{\mathcal{D}}} \mathcal{D}(f, 1) \\ &= \mathrm{lan}_{\mathcal{D}(1,f)} \text{よ}_{\mathcal{C}} \end{aligned}$$

while the left and right extensions can be computed as the extensions

$$\mathrm{lext}_f = \mathrm{lan}_{\text{よ}_{\mathcal{C}}} \mathcal{D}(1, f) \qquad \mathrm{rext}_f = \mathrm{ran}_{\text{よ}_{\mathcal{C}}} \mathcal{D}(1, f)$$

The formalism of extension and restrictions, applied to sieves, will be the common thread of the two parts of this work.

Extension of sieves

For a sieve $S \rightarrowtail よ_c$ on an object of $\mathcal{C}$, the left extension $\mathrm{lext}_f S$ is a sieve on $f(c)$ and can be computed at any object d as the coend

$$\mathrm{lext}_f(S)(d) = \int^{c' \in \mathcal{C}} D(d, f(c')) \times S(c')$$

which is exactly the set of arrows factorizing through the image of S

$$\left\{ v : d \to f(c) \;\middle|\; \begin{array}{ccc} d & \xrightarrow{\;v\;} & f(c) \\ {\scriptstyle\exists}\,\downarrow & \nearrow_{f(u)} & \\ f(c') & & \end{array} \quad \text{for some } u : c' \to c \in S(c') \right\}$$

Dually, for a functor $f : \mathcal{C} \to \mathcal{D}$ and a sieve $R \rightarrowtail \text{よ}_d$, the restriction $\text{rest}_f R$ can be computed at c' in $\mathcal{C}$ through the coend formula

$$\text{rest}_f R(c') = \int^{d' \in \mathcal{D}} R(d') \times \mathcal{D}(f(c'), d')$$

which is exactly the set of arrow from $f(c')$ in $\mathcal{D}$ that lie in R:

$$\left\{ v : f(c') \to d \;\middle|\; \begin{array}{ccc} f(c') & \xrightarrow{\;v\;} & d \\ {\scriptstyle \exists}\downarrow & \nearrow_{v'} & \\ d' & & \end{array} \quad \text{for some } v' : d' \to d \in R(d) \right\}$$

In the case where d is of the form $f(c)$, $\text{rest}_f R$ is a subobject of $\text{rest}_f \text{よ}_{f(c)}$.

However, beware that $\text{rest}_f \text{よ}_{f(c)} \simeq \mathcal{D}(f, f(c))$, which is not a representable on $\mathcal{C}$: hence $\text{rest}_f R$ is not itself yet a sieve on $\mathcal{C}$. However, it locally is !

Restriction of sieves

There is a canonical element $\nu_c : よ_c \to \mathrm{rest}_f\, よ_{f(c)}$ given by the canonical 2-cell $\nu : よ_{\mathcal{C}} \Rightarrow \mathcal{D}(f,1)f$ associated with the nerve of f.

Then one can consider the pullback presheaf

$$\begin{array}{ccc} f^{-1}(R) & \longrightarrow & \mathrm{rest}_f R \\ \downarrow & \lrcorner & \downarrow \\ よ_c & \xrightarrow[\nu_c]{} & \mathcal{D}(f, f(c)) \end{array}$$

which is now a subobject of $よ_c$ and corresponds to the sieve on c given by the set

$$\left\{ u : c' \to c \mid \begin{array}{ccc} f(c') & \xrightarrow{f(u)} & f(c) \\ \exists \downarrow & \nearrow_{v'} & \\ d' & & \end{array} \quad \text{for some } v' : d' \to f(c) \in R(d) \right\}$$

Cover-preservation and morphisms of sites

Definition

A functor between sites $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is said to be *cover-preserving* if for any J-covering sieve $S \rightarrowtail \text{よ}_c$ in $J(c)$, the sieve generated in $\mathcal{D}$ from arrows of the forms $f(u) : f(c') \to f(c)$ is K-covering.

Or using the extension functors, this rephrases as:

Definition

A functor between sites $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is *cover-preserving* if for any J-covering sieve $S \rightarrowtail \text{よ}_c$ in $J(c)$, the sieve $\mathrm{lext}_f S$ is in $K(f(c))$.

In general cover-preservation is combined with with flatness to produce a convenient notion of functor between sites:

Definition

A *morphism of sites* is a flat functor that is cover-preserving.
If $\mathcal{E}$ is a topos, a functor $(\mathcal{C}, J) \to \mathcal{E}$ is said to be *J-continuous* if it is flat and defines a morphism of sites into $(\mathcal{E}, J_{can})$ where J_{can} is the canonical topology on $\mathcal{E}$.

We will denote as $\mathbf{Sit}^\flat$ the 2-category of sites, morphisms of sites and transformations between them.

Geometric morphism from a morphism of sites

A morphism of sites $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\widehat{f}$ whose inverse image is constructed as the left Kan extension

$$\widehat{f}^* = \mathrm{lan}_{\mathfrak{y}_{(\mathcal{C},J)}} \mathfrak{y}_{(\mathcal{D},K)} f$$

where $\mathfrak{y}_{(\mathcal{C},J)} : \mathcal{C} \to \widehat{\mathcal{C}}_J$ and $\mathfrak{y}_{(\mathcal{D},K)} : \mathcal{D} \to \widehat{\mathcal{D}}_K$ are the embedding into sheaves.

Moreover the inverse image part is also related to the left adjoint lext_f as follows

$$\begin{array}{ccc} \widehat{C} & \xrightarrow{\mathrm{lext}_f} & \widehat{D} \\ {\scriptstyle \mathrm{i}_J}\big\uparrow & & \big\downarrow{\scriptstyle \mathfrak{a}_K} \\ \widehat{\mathcal{C}}_J & \xrightarrow[\widehat{f}^*]{} & \widehat{\mathcal{D}}_K \end{array}$$

This defines a pseudofunctor

$$(\mathbf{Sit}^{\flat})^{\mathrm{op}} \xrightarrow{\mathbf{Sh}} \mathbf{Top}$$

Definition

A functor between sites $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is said to be *cover-reflecting*, or also to be a *comorphism of sites*, if for any d in $\mathcal{D}$ and any J-covering sieve S on $f(d)$, there is a K-covering sieve R such that $f(v)$ is in S for all $u \in R$.

Again, this can be rephrased, this time with the restriction functor:

Definition

A functor between sites $f : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is *cover-reflecting* if for any d in $\mathcal{D}$ and any J-covering sieve S on $f(d)$, the restricted sieve $f^{-1}(S)$ on d is K-covering.

(Equivalently, if S contains a sieve $\text{lext}_f R$ for R a K-covering sieve on d.)

We will denote as $\mathbf{Sit}^\sharp$ the 2-category of sites, comorphisms of sites and transformations between them.

Geometric morphism from comorphism of sites

For a comorphism of site $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$, denote as $A_F : (\mathcal{C}, J) \to \widehat{\mathcal{D}}_K$ the composite $\mathfrak{a}_K \mathcal{C}(F, 1)$.

Then A_F is a J-continuous flat functor, and it induces a geometric morphism $\mathbf{Sh}(A_F) = C_F : \widehat{\mathcal{D}}_K \to \widehat{\mathcal{C}}_J$ whose inverse image is the composite

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{\mathrm{rest}_F} & \widehat{\mathcal{D}} \\ {\scriptstyle \mathfrak{i}_J}\uparrow & & \downarrow{\scriptstyle \mathfrak{a}_K} \\ \widehat{\mathcal{C}}_J & \underset{C_F^*}{\dashrightarrow} & \widehat{\mathcal{D}}_K \end{array}$$

This defines a pseudofunctor

$$(\mathbf{Sit}^{\sharp})^{\mathrm{co}} \xrightarrow{C} \mathbf{Top}$$

Mixing morphisms and comorphisms ?

Now one may ask whether morphisms and comorphisms may be mixed altogether into a same 2-dimensional structure:

$$\begin{array}{ccccc} (\mathbf{Sit}^{\flat})^{\mathrm{op}} & \dashrightarrow & ? & \dashleftarrow & (\mathbf{Sit}^{\sharp})^{\mathrm{co}} \\ & \searrow_{\mathbf{Sh}} & \downarrow & \swarrow_{C} & \\ & & \mathbf{Top} & & \end{array}$$

The problem is that morphisms of sites and comorphisms of sites do not compose with each other. But there are two way to fix this:

- either to arrange them into a *double category*
- either by jointly generalizing them into a notion of *distributors* between sites.

A double category of sites

Double categories

Definition (Ehresmann [3])

A *double category* $\mathbb{D}$ is the data of

- a class of *objects* $C, D \ldots$
- a class of *horizontal 1-cells* $f : C \to D$
- a class of *vertical 1-cells* $F : C \mathrel{\mapstochar\mkern-4mu\longrightarrow} D$
- a class of *double cells* of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle H}\big\downarrow & \phi & \big\downarrow{\scriptstyle K} \\ C & \xrightarrow[g]{} & D \end{array}$$

subject to the following axioms:

- each object C admits both a horizontal and vertical identity 1_C and id_C
- horizontal arrows from a category $h\mathbb{D}$ with identities given by horizontal identities
- vertical arrows from a category $v\mathbb{D}$ with identities given by vertical identities
- double cells paste horizontally and vertically
- pasting are subject to suited interchange and identity rules.

Double functors

Definition

A *double functor* between double categories $\mathbb{F} : \mathbb{C} \to \mathbb{D}$ is the data of

- a function $F : \mathrm{Ob}_{\mathbb{C}} \to \mathrm{Ob}_{\mathbb{D}}$ at the level of objects
- a horizontal component $h\mathbb{F} : h\mathbb{C} \to h\mathbb{D}$
- a vertical component $v\mathbb{F} : v\mathbb{C} \to v\mathbb{D}$
- and for every double cell in $\mathbb{C}$ a double cell in $\mathbb{D}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle H}\big\downarrow & \phi & \big\downarrow{\scriptstyle K} \\ C & \xrightarrow[g]{} & D \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{h\mathbb{F}f} & FB \\ {\scriptstyle v\mathbb{F}H}\big\downarrow & \mathbb{F}\phi & \big\downarrow{\scriptstyle v\mathbb{F}K} \\ FC & \xrightarrow[h\mathbb{F}g]{} & FD \end{array}$$

which are moreover to the suited coherences relative to pasting and identities.

Examples

The point with double categories is that they allow to consider 2-cells between two classes of morphisms without requiring them to compose altogether.

Example

The ur-example: the double category **Dist** whose

- objects are categories
- horizontal maps are functors
- vertical maps are *distributors* (a.k.a. *profunctors*)
- double cells are natural transformations

But there is a simpler example:

Example

For any 2-category $\mathcal{K}$, there is a *lax quintet* (resp. oplax quintet) double category $\mathcal{K}^{\square}_{\mathbf{lax}}$ (resp. $\mathcal{K}^{\square}_{\mathbf{oplax}}$) whose objects are those of $\mathcal{K}$, vertical and horizontal morphisms are any morphisms in $\mathcal{K}$, and double cells are lax (resp. oplax) squares.

(In fact $\mathcal{K}^{\square}_{\mathbf{lax}}$ and $\mathcal{K}^{\square}_{\mathbf{oplax}}$they are the same thing up to a transposition duality)

Definition

We define the double category $\mathbf{Sit}^{\natural}_{\mathbf{lax}}$ as having as objects (small generated) sites, as horizontal arrows morphisms of sites, as vertical arrows comorphisms of sites, and as a double cell

$$\begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\big\downarrow\!\!\!\!+ & \boxed{\phi} & +\!\!\!\!\big\downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array}$$

a lax squares as below, with f, h morphisms of sites and G, K comorphisms of sites:

$$\begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\big\downarrow & \overset{\phi}{\swarrow} & \big\downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array}$$

Constructing a double sheafification functor

We want to construct a sheafification double functor

- whose horizontal component is the pseudofunctor **Sh** form morphisms
- whose vertical component is the pseudofunctor C for comorphisms

For double cells, we use the canonical constructions of *mates*: suppose one has a 2-cell of the following form:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle k} \\ \mathcal{C} & \xrightarrow[h]{} & \mathcal{D} \end{array}$$

Such a square induces a cross-adjoint square constructed from the composite 2-cell

$$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\mathrm{lext}_f} & \widehat{\mathcal{B}} \\ {\scriptstyle \mathrm{rest}_g}\uparrow & \overset{\overline{\phi}}{\searrow} & \uparrow{\scriptstyle \mathrm{rest}_k} \\ \widehat{\mathcal{C}} & \xrightarrow[\mathrm{lext}_h]{} & \widehat{\mathcal{D}} \end{array} \qquad \begin{array}{ccc} \mathrm{lext}_f\ \mathrm{rest}_g & \xRightarrow{\eta_h} & \mathrm{lext}_f\ \mathrm{rest}_g\ \mathrm{rest}_h\ \mathrm{lext}_h \\ {\scriptstyle \overline{\phi}}\Downarrow & & \Downarrow{\scriptstyle \mathrm{rest}_\phi} \\ \mathrm{rest}_k\ \mathrm{lext}_h & \xLeftarrow[\epsilon_f]{} & \mathrm{lext}_f\ \mathrm{rest}_f\ \mathrm{rest}_k\ \mathrm{lext}_h \end{array}$$

Sheafification of double-cells

If now one has sites $(\mathcal{A}, M)$, $(\mathcal{B}, L)$, $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, related through a lax square

$$\begin{array}{ccc}
(\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\
{\scriptstyle G}\big\downarrow & \overset{\phi}{\swarrow\!\!\!\!\swarrow} & \big\downarrow{\scriptstyle K} \\
(\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K)
\end{array}$$

where f, h are morphisms of sites and G, K comorphisms of sites respectively, then the sheafification of the previous 2-cell along $\mathfrak{a}_L : \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}_L$ produces a 2-cell corresponding to a 2-cell between geometric morphisms

$$\begin{array}{ccc}
\widehat{\mathcal{A}}_M & \xrightarrow{\widehat{f}^*} & \widehat{\mathcal{B}}_L \\
{\scriptstyle C_G^*}\big\uparrow & \underset{\widehat{\phi}^*}{\searrow\!\!\!\!\searrow} & \big\uparrow{\scriptstyle C_K^*} \\
\widehat{\mathcal{C}}_J & \xrightarrow[\widehat{h}^*]{} & \widehat{\mathcal{D}}_K
\end{array}
\qquad\qquad
\begin{array}{ccc}
\widehat{\mathcal{A}}_M & \xleftarrow{\widehat{f}} & \widehat{\mathcal{B}}_L \\
{\scriptstyle C_G}\big\downarrow & \underset{\widehat{\phi}}{\searrow\!\!\!\!\searrow} & \big\downarrow{\scriptstyle C_K} \\
\widehat{\mathcal{C}}_J & \xleftarrow[\widehat{h}]{} & \widehat{\mathcal{D}}_K
\end{array}$$

But this is a 2-cell in the lax quintet double category $\mathbf{Top}^{\square}_{\mathbf{lax}}$ of Grothendieck topoi!

Sheafification as a double functor

Theorem

One has a horizontal contravariant, vertically covariant double functor and join full-on-objects-embeddings of the categories of sites with morphisms and comorphisms as the horizontal and vertical categories respectively:

$$\begin{array}{ccccc} (\mathbf{Sit}^{\flat})^{\mathrm{op}} & \xrightarrow{h} & \mathbf{Sit}^{\natural}_{\mathbf{lax}} & \xleftarrow{v} & (\mathbf{Sit}^{\sharp})^{\mathrm{co}} \\ & {\scriptstyle \mathbf{Sh}}\searrow & \downarrow & \swarrow{\scriptstyle C} & \\ & & \mathbf{Top}^{\square}_{\mathbf{lax}} & & \end{array}$$

In fact one could also define a double category $\mathbf{Sit}^{\natural}_{\mathbf{oplax}}$ with oplax squares, and construct a double functor $\mathbf{Sh}_{\mathbf{oplax}}$.

A way to fix this duplication of double cells would be to work at the level of the double category $\mathbf{Top}^{\square}_{\mathbf{relax}}$ where lax and oplax squares are subsumed by squares filled by a *natural relation*.

Why a double-categorical structure ?

One may wonder why morphisms and comorphisms arrange in a double category, and whether this double category is an instance of a special family of double categories.

The better known relational double categories as double categories of profunctors have a quite different flavour as they are less symmetrical.

Here horizontal and vertical maps are two classes of functors with dual properties, rather than a class of functors and a class of relations generalizing them.

This reminds a more symmetric kind of double categories, those that arise as double categories of (co)algebras with lax and colax morphisms for (co)monads !

Definition

Let $\mathcal{K}$ be a 2-category and $T : \mathcal{K} \to \mathcal{K}$ a copointed endo-2-functor, that is, equipped with a strict natural transformation $\varepsilon : T \Rightarrow 1_{\mathcal{K}}$. A *co-algebra* for (T, ε) is the data of a pair (C, γ) with C in $\mathcal{K}$ and α a section of the counit

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & TC \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon_C} \\ & & C \end{array}$$

A *lax* (resp. *colax*) *morphism* of co-algebras $(C, \gamma) \to (D, \delta)$ is a pair (f, ϕ) with $f : C \to D$ in $\mathcal{K}$ and ϕ a 2-cell as on the left (resp. on the right)

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \gamma}\downarrow & \overset{\phi}{\Longrightarrow} & \downarrow{\scriptstyle \delta} \\ TC & \xrightarrow[Tf]{} & TD \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \gamma}\downarrow & \overset{\phi}{\Longleftarrow} & \downarrow{\scriptstyle \delta} \\ TC & \xrightarrow[Tf]{} & TD \end{array}$$

whose pasting along the naturality square of the pointer is an identity 2-cell.

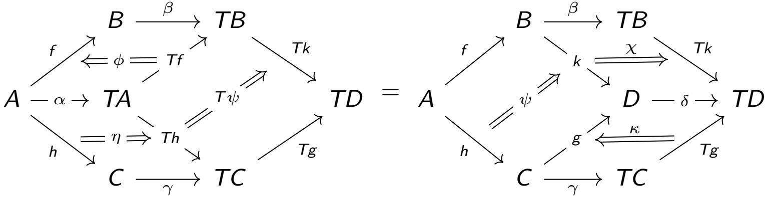
Double category of co-algebras, lax and colax morphisms

Proposition (Paré-Grandis [4])

*For any copointed endo-2-functor T, one can form a double category T-**coAlg** of strict co-algebras, lax morphisms as horizontal cells, colax morphisms as vertical cells, and as 2-cell, the lax squares of the form*

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{(f,\phi)} & (B,\beta) \\ {\scriptstyle (h,\eta)}\big\downarrow & \overset{\psi}{\Longrightarrow} & \big\downarrow{\scriptstyle (k,\chi)} \\ (C,\gamma) & \xrightarrow[(g,\kappa)]{} & (D,\delta) \end{array}$$

The double cells of this double category consist hence in 2-cells $\psi : gh \Rightarrow kf$ intertwinning the lax and colax morphism structures in the following coherence



Some observations on Grothendieck coverages

Recall that for a Grothendieck coverage on a category $\mathcal{C}$, the set $J(c)$ of J-covering sieves $S \rightarrowtail\, よ_c$ on an object c form a subposet $J(c)$ of the poset of subobject $\mathrm{Sub}_{\widehat{\mathcal{C}}}\, よ_c$.

Moreover the poset $J(c)$ is up-closed and contains in particular the maximal sieve.

For each category $\mathcal{C}$ and each c in $\mathcal{C}$, define $\mathbb{F}_{\mathcal{C}}(c)$ the poset of filters of $\mathrm{Sub}_{\widehat{\mathcal{C}}}\, よ_c$ that is, is objects are filters $F \hookrightarrow \mathrm{Sub}_{\widehat{\mathcal{C}}}$, upsets containing the top element $よ_c$.

Now a morphism $u : c \to c'$ defines at each $S \rightarrowtail\, よ_{c'}$ a pullback sieve $u^*S \to\, よ_c$.

This defines a morphism of posets $u^* : \mathrm{Sub}_{\widehat{\mathcal{C}}}(c') \to \mathrm{Sub}_{\widehat{\mathcal{C}}}(c)$. If now F is a filter of $\mathrm{Sub}_{\widehat{\mathcal{C}}}(c')$, the inverse image

$$(u^*)^{-1}(F) = \{R \rightarrowtail\, よ_{c'} \mid u^*R \in F\}$$

is a filter of $\mathrm{Sub}_{\widehat{\mathcal{C}}}(c)$. Hence we have a morphism of posets

$$\mathbb{F}_{\mathcal{C}}(c) \xrightarrow{(u^*)^{-1}} \mathbb{F}_{\mathcal{C}}(c')$$

The category $\mathbb{S}\mathcal{C}$

Definition

Define for each category $\mathcal{C}$ the category $\mathbb{S}\mathcal{C}$ as having:

- as objects pairs (c, F) with c an object of $\mathcal{C}$ and F a filter of $\mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_c$
- as morphisms $(c, F) \to (c', F')$ morphisms $u : c \to c'$ such that

$$F' \leq (u^*)^{-1}(F)$$

which amounts to asking that for any $R \rightarrowtail \text{よ}_{c'}$ in F' one has $u^*R \in F$, as visualized by the condition that u^* restricts between F' and F seen as subposets

$$\begin{array}{ccc} \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_{c'} & \xrightarrow{u^*} & \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_c \\ \uparrow & & \uparrow \\ F' & \dashrightarrow & F \end{array}$$

This category is fibered over $\mathcal{C}$ with posetal fibers $\mathbb{F}_{\mathcal{C}}(c)$ at each c

$$\mathbb{S}\mathcal{C} \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C}$$

2-Functoriality of $\mathbb{S}$

For a functor $f : \mathcal{C} \to \mathcal{D}$, one can define for each $S \rightarrowtail よ_c$ in $\mathcal{C}$ the sieve

$$\text{lext}_f S = \{v : d \to f(c) \mid \exists u : c' \to c \in S \text{ such that } v \leq f(u)\}$$

For a given filter F of $\text{Sub}_{\widehat{\mathcal{C}}} よ_c$, one can consider the filter $\text{lext}_f[F]$ generated from the set of sieves of the form $\text{lext}_f(S)$ for S in F. This filter contains all sieves $R \to よ_{f(c)}$ that contain a sieve of the form $\text{lext}_f(S)$ for S in F.

If now one has a morphism $u : c \to c'$ in $\mathcal{C}$, then one has a morphism in $\mathbb{S}\mathcal{D}$

$$(f(c),\ \text{lext}_f[F]) \xrightarrow{fu} (f(c'),\ \text{lext}_f[F'])$$

Hence one just has to defines the functor $\mathbb{S}f : \mathbb{S}\mathcal{C} \to \mathbb{S}\mathcal{D}$ sending (c, F) to the pair $(f(c),\ \text{lext}_f[F])$.

The copointed endo-2-functor $\mathbb{S}$

Hence this construction is functorial on **Cat**: we have an endo-2-functor

$$\mathbf{Cat} \xrightarrow{\mathbb{S}} \mathbf{Cat}$$

Moreover this endofunctor is copointed through the projections $\pi_{\mathcal{C}} : \mathbb{S}\mathcal{C} \to \mathcal{C}$, whose naturality produce morphisms of fibrations

$$\begin{array}{ccc} \mathbb{S}\mathcal{C} & \xrightarrow{\mathbb{S}f} & \mathbb{S}\mathcal{D} \\ {\scriptstyle \pi_{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{D}}} \\ \mathcal{C} & \xrightarrow[f]{} & \mathcal{D} \end{array}$$

In fact this extends nicely to an endo-2-functor on **Lex**.

Proposition

A co-algebra structure on $\mathcal{C}$ for $\mathbb{S}$ is a coverage on $\mathcal{C}$.

A co-algebra defines a section $J : \mathcal{C} \to \mathbb{S}\mathcal{C}$ of $\pi_{\mathcal{C}}$. This means that for any c the object $J(c)$ is of the form $(c, J(c))$ where $J(c)$ is a filter of $\mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_c$, that is, consists of a collection of sieve such that よ_c is in $J(c)$, $J(c)$ is up-closed for the inclusion.

Moreover functoriality says that for any $u : c \to c'$, one has a restriction

$$
\begin{array}{ccc}
J(c) & \hookrightarrow & \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_c \\
\uparrow & & \uparrow u^* \\
J(c') & \hookrightarrow & \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_{c'}
\end{array}
$$

expressing that for any R in $J(c')$ the pullback sieve u^*R is in $J(c)$.

This is exactly what a coverage is!

Morphisms as lax morphisms of co-algebras

Proposition

A functor $f : \mathcal{C} \to \mathcal{D}$ is a lax morphism of co-algebras iff it is cover-preserving

Suppose that one has a 2-cell

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle J}\big\downarrow & \overset{\phi}{\Longrightarrow} & \big\downarrow{\scriptstyle K} \\ \mathbb{S}\mathcal{C} & \underset{\mathbb{S}f}{\longrightarrow} & \mathbb{S}\mathcal{D} \end{array}$$

This consists for each c in $\mathcal{C}$ of a morphism $(f(c), \ \mathrm{lext}_f[J(c)]) \to (f(c), K(f(c)))$, with the same supporting object $f(c)$: but as $\mathbb{S}\mathcal{D}$ has posetal fibers, the existence of such a 2-cell amounts to an inequality between subobjects

$$\mathrm{lext}_f[J(c)] \leq K(f(c))$$

which means that if a sieve R contains the image of a sieve of $J(c)$, then it is in $K(f(c))$. Hence in particular for any S in $J(c)$, $\mathrm{lext}_f S$ is in $K(f(c))$, which makes f a cover-preserving functor $(\mathcal{C}, J) \to (\mathcal{D}, K)$.

Comorphisms as colax morphisms of co-algebras

Proposition

A functor $f : \mathcal{C} \to \mathcal{D}$ is a colax morphism of co-algebras iff it is cover-reflecting

Suppose that one has a 2-cell

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ f\ } & \mathcal{D} \\ {\scriptstyle J}\big\downarrow & \overset{\phi}{\Longleftarrow} & \big\downarrow{\scriptstyle K} \\ \mathbb{S}\mathcal{C} & \xrightarrow[\ \mathbb{S}f\ ]{} & \mathbb{S}\mathcal{D} \end{array}$$

This consists for each c in $\mathcal{C}$ of a morphism $(f(c), K(f(c))) \to (f(c),\ \mathrm{lext}_f[J(c)])$, with the same supporting object $f(c)$: but as $\mathbb{S}\mathcal{D}$ has posetal fibers, the existence of such a 2-cell amounts to an inequality between subobjects

$$K(f(c)) \leq \ \mathrm{lext}_f[J(c)]$$

which means any sieve R in $K(f(c))$ contains a sieve of the form $\mathrm{lext}_f S$ for some S in $J(c)$, which makes f a cover-reflecting functor $(\mathcal{C}, J) \to (\mathcal{D}, K)$.

Further double categorical aspects and questions

There are still several questions regarding this double categorical approach:

- How localness axiom of Grothendieck topologies can be expressed with $\mathbb{S}$?
- Is $\mathbb{S}$ actually a comonad, and so, what is the topological meaning of the additional coherence condition of its algebras ?
- The exact interaction with flatness should be clarified to exhibit morphisms of sites as lax morphisms of co-algebras.
- There are dualities as in [2] allowing to construct morphisms from comorphisms and conversely; in particular there is a comma constructions which is a *tabulator* in $\mathbf{Sit}^{\natural}$. What can be said about other double categorical constructions ?
- Those duality produce double cells akin to *conjoint cells* in $\mathbf{Sit}^{\natural}$, but they become *companion pairs* in $\mathbf{Top}$.
- It is known since [6] that $\mathbf{Top}$ is bilocalization of $\mathbf{Sit}^{\flat}$ at *dense morphisms of sites*. We expect this to extend to a double localization of $\mathbf{Sit}^{\natural}$.

Continuous distributors

Joint generalization of morphisms and comorphisms ?

There is another way to mix morphisms and comorphisms, this time as two instances of a same notion of 1-cells.

It also relates to the imperfect correspondence of (co)morphisms of sites with geometric morphisms:

For two fixed sites $(\mathcal{C}, J)$, $(\mathcal{D}, K)$, there are geometric morphisms $\widehat{\mathcal{D}}_K \to \widehat{\mathcal{C}}_J$ that are not induced from morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, for the inverse image may not restrict to a functor.

But it always restricts to *a distributor*, and we may ask what flatness and continuity mean for distributors.

Distributors and heteromorphisms

Definition

A *distributor* (a.k.a. *profunctor*) $H : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $\mathcal{D}^{op} \times \mathcal{C} \to \mathbf{Set}$.

For a distributors $H : \mathcal{C} \nrightarrow \mathcal{D}$, élements of the sets $H(d, c)$ will be called *heteromorphisms* and denoted as generalized morphisms across categories

$$b \overset{a}{\rightsquigarrow} c$$

For $v : d' \to d$ and $u : c \to c'$, $H(v, c)$ and $H(d, u)$ act as pre/post composition.

Example

Any functor $F : \mathcal{C} \to \mathcal{D}$ induces two *representable* distributors:

- the right representable $\mathcal{D}(1, F) : \mathcal{C} \nrightarrow \mathcal{D}$ sending (d, c) to $\mathcal{D}(d, F(c))$,
- the left representable $\mathcal{D}(F, 1) : \mathcal{D} \nrightarrow \mathcal{C}$ sending (c, d) to $\mathcal{D}(F(c), d)$.

The composite of distributors is computed at a pair (e, c) as the coend

$$H \otimes K(e, c) = \int^{d \in \mathcal{D}} H(d, c) \times K(e, d)$$

Extension along distributors

Distributors induce functors between presheaf categories:

$$\begin{aligned}\mathbf{Dist}[\mathcal{C}, \mathcal{D}] &= \mathbf{Cat}[\mathcal{D}^{\mathrm{op}} \times \mathcal{C}, \mathbf{Set}] \\ &\simeq \mathbf{Cat}[\mathcal{C}, \widehat{\mathcal{D}}] \\ &\simeq \mathbf{coCont}[\widehat{\mathcal{C}}, \widehat{\mathcal{D}}]\end{aligned}$$

where a distributor $H : \mathcal{C} \nrightarrow \mathcal{D}$ is sent successively to

- the functor $\widehat{H} : \mathcal{C} \to \widehat{\mathcal{D}}$ sending c to the presheaf $H(-, c)$ on $\mathcal{D}$
- then to the cocontinuous functor $\mathrm{lext}_H : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ obtained through the left Kan extension $\mathrm{lext}_H = \mathrm{lan}_{よ_{\mathcal{C}}} \widehat{H}$ which is computed at an object d as the coend:

$$\mathrm{lext}_H(X)(d) = \int^{c \in \mathcal{C}} H(d, c) \times X(c)$$

Moreover lext_H possesses a right adjoint $\mathrm{rest}_H = \mathrm{lan}_H よ_{\mathcal{D}}$ computed as

$$\mathrm{rest}_H(Y)(c) = \widehat{\mathcal{D}}[\widehat{H}(c), Y]$$

Extension along representable

The extensions interact nicely with representables, in a way that will help unify cover-preservation and cover-reflection:

Proposition

For $f : \mathcal{C} \rightarrow \mathcal{D}$, the right representable $\mathcal{D}(1, f) : \mathcal{C} \nrightarrow \mathcal{D}$ corresponds with the right nerve and its left extension coincides with the left extension functor along f:

$$\mathrm{lext}_{\mathcal{D}(1,f)} = \mathrm{lext}_f$$

But we have a funny twist for the other representable:

Proposition

For $f : \mathcal{D} \rightarrow \mathcal{C}$, the left representable $\mathcal{C}(f, 1) : \mathcal{C} \nrightarrow \mathcal{D}$ corresponds with the left nerve, but its left extension coincides with the restriction functor along f

$$\mathrm{lext}_{\mathcal{C}(f,1)} = \mathrm{rest}_f$$

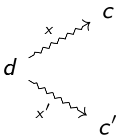
As we will see, those formulas will play an important role when unifying cover-preserving and cover-lifting condition into a same condition for distributors.

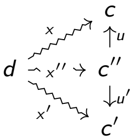
Definition (Benabou [1])

Then a distributor $H : \mathcal{C} \nrightarrow \mathcal{D}$ is said to be *representably flat* if for each d the category $d \downarrow H$ is cofiltered.

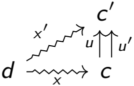
This generalizes exactly the usual notion of representable flatness:

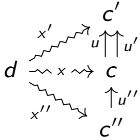
- $d \downarrow H$ is non empty, so there is an heteromorphism $d \rightsquigarrow c$
- for any diagram on the left is the is a span as on the right





- and for any diagram as below there is $u'' : c'' \to c$ with $uu'' = u'u''$ and x'' such that $u''(x'') = x$:





Flat distributors beget geometric morphisms between presheaf topoi

Proposition (Benabou [1])

For a distributor $H : \mathcal{C} \nrightarrow \mathcal{D}$ the following are equivalent

- *H is representably flat*
- *the left extension $\operatorname{lext}_H : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is lex*
- *the pair $\widehat{H} = (\operatorname{lext}_H, \operatorname{rest}_H) : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is a geometric morphism*

Proposition

Conversely, any geometric morphism $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ is induced by a flat distributor.

Indeed, f is induced by a flat functor $f : \mathcal{C} \to \widehat{\mathcal{D}}$ and one just take $H_f : \mathcal{C} \nrightarrow \mathcal{D}$ sending (d, c) to the evaluation $f(c)(d)$.

Proposition

The pseudofunctor $\widehat{(-)} : \mathbf{FlatDist}^{\mathrm{op}} \to \mathbf{Top}$ is pseudofully faithful, that is, for any $\mathcal{C}, \mathcal{D}$ we have an equivalence of categories

$$\mathbf{FlatDist}[\mathcal{C}, \mathcal{D}] \simeq \mathbf{Top}[\widehat{\mathcal{D}}, \widehat{\mathcal{C}}]$$

Continuity conditions ?

Continuity conditions for distributors have been not so much investigated until now, except in term of *bidense* morphisms in Johnstone and Wraith "Algebraic theories in toposes" [5].

Here we would like to give nice joint generalization of cover-preservation, cover-reflection, and continuity for distributors.

To extract such a notion, let us re-express cover-preservation and reflection in terms of heteromorphisms associated to representable distributors.

Extension of sieves along distributors

Let $H : \mathcal{C} \nrightarrow \mathcal{D}$ be a distributor between sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$.
Then the left extension of a sieve $S \rightarrowtail よ_c$ returns at any object d the coend

$$\mathrm{lext}_H(S)(d) = \int^{c' \in \mathcal{C}} H(d, c') \times S(c')$$

which is exactly the set of heteromorphisms factorizing through S

$$\left\{ a \in H(d,c) \mid \exists u' : c' \to c \in S \text{ and } \exists a' \in H(d,c') \text{ such that } \quad \begin{array}{ccc} & & c' \\ & \overset{a'}{\nearrow} & \big\downarrow{\scriptstyle u'} \\ d & \underset{a}{\rightsquigarrow} & c \end{array} \right\}$$

This exhibits $\mathrm{lext}_H(S)$ as a subobject of the presheaf $\widehat{H}(c)$ in $\widehat{\mathcal{D}}$.

Though this is not properly speaking a sieve, it is *locally* a sieve !

Indeed, by Yoneda lemma, an heteromorphism $a \in H(d,c)$ is a transformation $a : よ_d \to \widehat{H}(c)$, along which the pullback of the extension produces a sieve on d

$$\begin{array}{ccc} a^* \ \mathrm{lext}_H S & \longrightarrow & \mathrm{lext}_H S \\ \downarrow & \lrcorner & \downarrow \\ よ_d & \xrightarrow[a]{} & \widehat{H}(c) \end{array}$$

Concretely, this object $a^* \ \mathrm{lext}_H S$ returns at an object d' of $\mathcal{D}$ the set

$$\left\{ v : d' \to d \mid \exists u : c' \to c \in S, \text{ and } \exists a' \in H(d',c') \text{ such that } \begin{array}{ccc} d' & \overset{a'}{\rightsquigarrow} & c' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u'} \\ d & \underset{a}{\rightsquigarrow} & c \end{array} \right\}$$

Now this defines a sieve on d in $\mathcal{D}$!

Cover-distributing property

Definition

A distributor $H : \mathcal{C} \nrightarrow \mathcal{D}$ between sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ will be said to be *cover-distributing* if for any pair (d, c), any heteromorphism $a \in H(d, c)$ and any J-covering sieve S on c, the sieve $a^* \ \mathrm{lext}_H(S)$ is K-covering on d.

This conditions subsumes both cover preservation and cover reflection:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be two sites; then:

- *a functor $f : \mathcal{C} \to \mathcal{D}$ is cover-preserving iff $\mathcal{D}(1, f) : \mathcal{C} \nrightarrow \mathcal{D}$ is cover-distributing*
- *a functor $f : \mathcal{D} \to \mathcal{C}$ is cover-reflecting iff $\mathcal{C}(f, 1) : \mathcal{C} \nrightarrow \mathcal{D}$ is cover-distributing.*

Cover-distribution subsumes cover-preservation and reflection

For the cover-preservation: use that $\mathrm{lext}_{\mathcal{D}(1,f)} = \mathrm{lext}_f$ and the fact that an heteromorphism $a \in \mathcal{D}(1, f(c))$ simply is a morphism $a : d \to f(c)$ in $\mathcal{D}$:

- if f is cover-preserving, for each J-covering sieve S the sieve $\mathrm{lext}_f S$ is K-covering, and so is $a^* \mathrm{lext}_f S$ so $\mathcal{D}(1, f)$ is cover-distributing.
- if $\mathcal{D}(1, f)$ is cover-distributing, apply the case $a = 1_{f(c)}$.

For the cover-reflection: an heteromorphism $a \in \mathcal{C}(f, 1)$ is a morphism $a : f(d) \to c$ in $\mathcal{C}$. Then for any J-sieve S on c, the pullback sieve a^*S is in $J(f(d))$, and then observe that $a^* \mathrm{lext}_{\mathcal{C}(f,1)} S = f^{-1}(a^*S)$:

- if f is cover-reflecting, each $f^{-1}(a^*S)$ must be K-covering
- if $\mathcal{C}(f, 1)$ is cover-distributing, then apply to the case $a = 1_{f(d)}$.

Combining cover-distribution and flatness

Definition

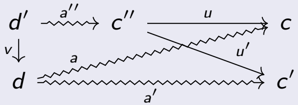
A distributor $H : (\mathcal{C}, J) \nrightarrow (\mathcal{D}, K)$ will be said to be (J, K)-*flat* if:

- H is cover-distributing
- For any d of $\mathcal{D}$, the following sieve is K-covering

$$\{v : d' \to d \mid \exists c \in \mathcal{C}, H(d', c) \neq \emptyset\}$$

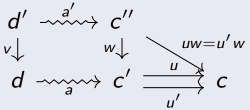
- For any c, c', d and $a \in H(d, c)$, $a' \in H(d, c')$, the following sieve is K-covering

$$\left\{ v : d' \to d \;\middle|\; \begin{array}{l} \exists u : c'' \to c, \\ \exists u' : c'' \to c' \\ \exists a'' \in H(d', c'') \end{array} \quad \text{such that} \right\}$$



- For any $u, u' : c' \rightrightarrows c$ in $\mathcal{C}$, d of $\mathcal{D}$ and $a \in H(d, c')$ with $H(d, u)(a) = H(d, u')(a)$, the following sieve is K-covering:

$$\left\{ v : d' \to d \;\middle|\; \begin{array}{l} \exists w : c'' \to c' \\ \exists a' \in H(d', c'') \end{array} \quad \text{such that} \right\}$$



Comparison with Johnstone-Wraith continuity

This definition is more or less equivalent to the following:

Definition (Johnstone-Wraith [5])

A distributor $\mathcal{C} \nrightarrow \mathcal{D}$ is *Johnstone-Wraith-flat* if for any finite diagram $(E_i)_{i \in I} : I \to \widehat{\mathcal{C}}$ and any J-bidense morphism $u : E \to \lim_{i \in I} E_i$, the induced map $\mathrm{lext}_H(u) : \mathrm{lext}_H(E) \to \mathrm{lext}_H(\lim_{i \in I} E_i)$ is K-bidense in $\widehat{\mathcal{D}}$.

Then we have an adjunction between homcategories:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be two small generated sites; then one has an adjunction between geometric morphisms and (J, K)-flat ditributors

$$\mathbf{Top}[\widehat{\mathcal{D}}_K, \widehat{\mathcal{C}}_J] \underset{\mathbb{H}}{\overset{\widehat{(-)}}{\rightleftarrows}} \mathbf{Dist}_{(J,K)}(\mathcal{C}, \mathcal{D}) \qquad \bot$$

where moreover the functor $\mathbb{H}$ is fully faithful.

Continuous distributors correspond to geometric morphisms

The hindrance to an equivalence is the problem that sheafification $\mathfrak{a}_K$ may identify distributors. This can be fixed by adding a condition of continuity:

Definition

A distributor $H : (\mathcal{C}, J) \nrightarrow (\mathcal{D}, K)$ will be said to be (J, K)-*continuous* if it is (J, K)-flat and satisfies that $\widehat{H} : \mathcal{C} \to \widehat{\mathcal{D}}$ takes values in the sheaf topos $\widehat{\mathcal{D}}_K$.

Denote as $\mathbf{ContDist}[(\mathcal{C}, J), (\mathcal{D}, K)]$ the category of (J, K)-continuous distributors.

Theorem

Any (J, K)-continuous distributor induces faithfully a J-continuous functor $\mathrm{lext}_H : \mathcal{C} \to \widehat{\mathcal{D}}_K$ so we have an equivalence

$$\mathbf{Top}[\widehat{\mathcal{D}}_K, \widehat{\mathcal{C}}_J] \simeq \mathbf{ContDist}[(\mathcal{C}, J), (\mathcal{D}, K)]$$

Reconciliating morphisms and comorphisms

Denote as $\mathbf{Sit}^{\rightsquigarrow}$ the bicategory of sites, continuous distributors and transformations.

Theorem

The left and right representable construction define pseudofunctors from $\mathbf{Sit}^{\flat}$ *and* $\mathbf{Sit}^{\sharp}$ *into* $\mathbf{Sit}^{\rightsquigarrow}$*, while the construction above defines a fully faithful embedding*

$$\begin{array}{ccccc} (\mathbf{Sit}^{\flat})^{\text{op}} & \xrightarrow{R} & (\mathbf{Sit}^{\rightsquigarrow})^{\text{op}} & \xleftarrow{L} & (\mathbf{Sit}^{\sharp})^{\text{co}} \\ & \searrow^{\mathbf{Sh}} & \downarrow \widehat{(-)} & \swarrow^{C} & \\ & & \mathbf{Top} & & \end{array}$$

Thank you for your attention !

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