2-categorical constructions on classifying toposes

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Toposes in Mondovì

Introduction

- In this presentation we provide logical descriptions of some fundamental constructions in the 2-category of Grothendieck toposes **GrTop**.
- More specifically, we describe geometric theories classified by some weighted limits of classifying toposes and morphisms between them induced by interpretations.
- As we shall see, the logical approach allows us to describe weighted limits of toposes in a simpler way than through the geometric perspective addressed in the literature.
- More specifically, in the literature some weighted limits of toposes are presented as categories of sheaves on some site $(\mathcal{C}, J)$, where J is a Grothendieck topology generated by some covering families in $\mathcal{C}$, i.e. the finest Grothendieck topology J for which these families are J-covering. In general, it is not simple to describe all the covering sieves for such a Grothendieck topology.

Example: comma toposes

Definition

The comma topos of a pair of geometric morphisms $f\colon \mathcal{F} \to \mathcal{E}$ and $g\colon \mathcal{G} \to \mathcal{E}$ is a topos (f/g) which is universally equipped with a square of the form

$$\begin{array}{ccc} (f/g) & \xrightarrow{\pi''} & \mathcal{G} \\ {\scriptstyle \pi'}\downarrow & \overset{}{\underset{\gamma}{\Longrightarrow}} & \downarrow{\scriptstyle g} \\ \mathcal{F} & \xrightarrow[f]{} & \mathcal{E} \end{array} .$$

- Under the assumption that $f\colon \mathcal{F} \to \mathcal{E}$ (resp. $g\colon \mathcal{G} \to \mathcal{E}$) is induced by a morphism of cartesian sites $p\colon (\mathcal{C}, J) \to (\mathcal{C}', J')$ (resp. $q\colon (\mathcal{C}, J) \to (\mathcal{C}'', J''))$ a presentation of the comma topos (f/g) as a category of sheaves on a site $(\mathcal{D}, J_{\mathcal{D}})$ was originally introduced by Deligne and subsequently studied by other mathematicians including Laumon, Abbes and Gros.

Example: comma toposes

- The category $\mathcal{D}$ is defined in the following way.

Objects: An object, denoted by $(d \to c \leftarrow e)$, consists of an object c in $\mathcal{C}$, an arrow $d \to p(c)$ in $\mathcal{C}'$ and an arrow $e \to q(c)$ in $\mathcal{C}''$.

Arrows: An arrow $(d' \to c' \leftarrow e') \to (d \to c \leftarrow e)$ consists of three arrows $c \to c'$, $d \to d'$ and $e \to e'$ in $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$, respectively, such that the obvious squares commute.

- The topology $J_{\mathcal{D}}$ is generated by covering families $\{(d_i \to c_i \leftarrow e_i) \to (d \to c \leftarrow e)\}_{i \in I}$ of the following three types
  1. $e_i = e$, $c_i = c$ for every i, and $\{d_i \to d\}_{i \in I}$ is J'-covering;
  2. $d_i = d$, $c_i = c$ for every i, and $\{e_i \to e\}_{i \in I}$ is J''-covering;
  3. I is the singleton $\{'\}$, $d' = d$ and the arrow $e' \to e \times_{q(c)} q(c')$ is an isomorphism.

- We provide a simpler presentation of the topos (f/g) using the logical perspective. More precisely, we assume that $f \colon \mathcal{F} \to \mathcal{E}$ (resp. $g \colon \mathcal{G} \to \mathcal{E}$) is induced by a morphism of syntactic sites and describe a theory classified by the topos (f/g).

The notion of classifying topos

Definition

A classifyng topos for a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathcal{E}_{\mathbb{T}}$ such that for every topos $\mathcal{H}$ there exists an equivalence of categories

$$\mathbf{GrTop}(\mathcal{H}, \mathcal{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-}\mathit{Mod}(\mathcal{H})$$

which is natural in $\mathcal{H}$.

Theorem (A. Joyal, G. Reyes and M. Makkai)

- *Every geometric theory $\mathbb{T}$ admits a classifying topos.*
- *Every topos $\mathbf{Sh}(\mathcal{C}, J)$ is the classifyng topos of some theory $\mathbb{T}$.*

Sketch

- *Let $\mathbb{T}$ be a geometric theory. The category $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of sheaves over the syntactic site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ of $\mathbb{T}$ is a classifying topos for $\mathbb{T}$.*
- *By Diaconescu's equivalence we have, for every Grothendieck topos $\mathcal{H}$, an equivalence of categories*

  $$\mathbf{GrTop}(\mathcal{H}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{H}),$$

  where $\mathbf{Flat}_J(\mathcal{C}, \mathcal{H})$ denotes the category of J-continuous flat functors.

Weighted limit of toposes

Definition

- *A weighted diagram of type $\mathcal{J}$ valued in the 2-category* **GrTop** *is a pair of (2)-functors (D, W) of the form:*

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{W} & \textbf{Cat} \\ {\scriptstyle D}\downarrow & & \\ \textbf{GrTop} & & \end{array} .$$

- *A weighted cone L over a weighted diagram (D, W) valued in the 2-category* **GrTop** *consists of a topos $\mathcal{H}$, which is called vertex of the cone, endowed with a pseudo-natural transformation $W \Rightarrow \textbf{GrTop}(\mathcal{H}, D(-))$.*
- *A* weighted limit *over a weighted diagram (D, W) of type $\mathcal{J}$ valued in* **GrTop** *is a cone L over (D, W) that satisfies the property to be universal, i.e. composition with L is, for every topos $\mathcal{H}$, one half of an equivalence of categories of the form*

$$\textbf{GrTop}(\mathcal{H}, \mathcal{E}) \cong \textbf{Cone}(\mathcal{H}, (D, W)),$$

where $\mathcal{E}$ denotes the vertex of L and $\textbf{Cone}(\mathcal{H}, (D, W))$ denotes the category of cones over the weighted diagram (D, W) with vertex $\mathcal{H}$.

GrTop has small and finite weighted limits

Lemma (M. Kelly, R. Street)

If a 2-category $\mathcal{A}$ has finite (resp. small) products, equalizers and cotensors with **2**, *then it has all finite (resp. small) weighted limits, i.e. those weighted limits whose type $\mathcal{J}$ is a finite category and the values of the weighting W are finite categories.*

Lemma (Theorem B4.1.4 [2])

The 2-category **GrTop** *has cotensors with* **2**.

Theorem ([2])

The 2-category of Grothendieck toposes has all finite (resp. small) weighted limits.

Interpretation of a theory into another one

Definition (O. Caramello)

1. *An interpretation of a geometric theory $\mathbb{T}$ into another geometric theory $\mathbb{T}'$ is a morphism of sites $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \to (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$.*
2. *A geometric theory $\mathbb{T}'$ is a geometric expansion of a geometric theory $\mathbb{T}$, if we can obtain $\mathbb{T}'$ form $\mathbb{T}$ by adding sorts, function symbols or relations symbols to the signature of $\mathbb{T}$ and geometric axioms over the resulting extended signature.*

Remark

Every geometric expansion $\mathbb{T}'$ of a theory $\mathbb{T}$ defines a canonical interpretation $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$.
Moreover, a morphism of sites $p\colon (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \to (\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$ induces a geometric morphism $p_{\mathbb{T}}^{\mathbb{T}'} : \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{\ p\ } & \mathcal{C}_{\mathbb{T}'} \\ {\scriptstyle l_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle l_{\mathbb{T}'}} \\ \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) & \xrightarrow{\ {p_{\mathbb{T}}^{\mathbb{T}'}}^{*}\ } & \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \end{array}$$

commutes.

The general construction

- Let (D, W) be a small weighted diagram of type $\mathcal{J}$ valued in the 2-category **GrTop**. We suppose that for every object j in $\mathcal{J}$ the topos $D(j)$ is the classifying topos of some geometric theory $\mathbb{T}_j$ and for every arrow $s : i \to j$ in $\mathcal{J}$ the geometric morphism $D(s) : \mathcal{E}_{\mathbb{T}_i} \to \mathcal{E}_{\mathbb{T}_j}$ is induced by an interpretation of $\mathbb{T}_j$ into $\mathbb{T}_i$, i.e. a morphism of sites $(\mathcal{C}_{\mathbb{T}_j}, J_{\mathbb{T}_j}) \to (\mathcal{C}_{\mathbb{T}_i}, J_{\mathbb{T}_i})$ between the respective syntactic sites.
- If we assume that the vertex of the weighted limit over some weighted diagram (D, W) as before is a classifying topos of some geometric theory $\mathbb{T}$, i.e. $Lim_{\mathcal{J}}(D, W) \simeq \mathcal{E}_{\mathbb{T}}$, then the equivalence of categories

$$\mathbb{T}\text{-}Mod(\mathcal{H}) \simeq \mathbf{GrTop}(\mathcal{H}, \mathcal{E}_{\mathbb{T}}) \simeq \mathbf{Cone}(\mathcal{H}, (D, W))$$

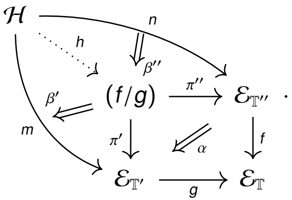
  allows us to regard, up to equivalence of categories, a $\mathbb{T}$-model in $\mathcal{H}$ as weighted cone over (D, W) with vertex $\mathcal{H}$.
- We describe a theory $\mathbb{T}$ classified by the vertex of (D, W) as a geometric expansion of every $\mathbb{T}_i$.

Some examples of (weighted) limits

- In the rest of this presentation we specialize and adapt this general construction to the notable cases of comma toposes, fibred products of toposes and small limits of toposes.
- In a forthcoming paper
  - we specialize and adapt this general construction to the cases of
    1. inserters of a pair of geometric morphisms,
    2. inverters of a geometric transformation,
    3. cotensors with a (small) category,
    4. equifiers of a pair of geometric transformations,
    5. descent objects in **GrTop**,
  - we provide a syntactic description of a general small or finite weighted limit of classifying toposes.

Comma toposes: from the universal property to the semantics

- Let $f\colon \mathcal{E}_{\mathbb{T}''} \to \mathcal{E}_{\mathbb{T}}$ and $g\colon \mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$ be geometric morphisms. The (1-dim)-universal property of the comma topos (f/g) states that for every topos $\mathcal{H}$ with a pair of geometric morphisms $m\colon \mathcal{H} \to \mathcal{E}_{\mathbb{T}'}$ and $n\colon \mathcal{H} \to \mathcal{E}_{\mathbb{T}''}$ endowed with a natural transformation $\omega\colon f \circ n \Rightarrow g \circ m$, there exists an unique, up to isomorphism, geometric morphism $h\colon \mathcal{H} \to (f/g)$ and a pair of geometric 2-isomorphisms $\beta'' : \pi'' \circ h \cong n$ and $\beta' : \pi' \circ h \cong m$ such that $\omega = (f \circ \beta'')(\alpha \circ h)(g \circ \beta'')$



Comma toposes: from the universal property to the semantics

- By the (1-dim)universal property of the comma topos (f/g) we deduce that a $\mathbb{T}'''$-model H in $\mathcal{H}$, where $\mathbb{T}'''$ is a theory classified by (f/g), is up to isomorphisms, a triplet (M, N, η), where M is a $\mathbb{T}'$-model in $\mathcal{H}$, N is a $\mathbb{T}''$-model in $\mathcal{H}$ and $\eta\colon N^f_{\mathbb{T}} \to M^g_{\mathbb{T}}$ is an homomorphism of $\mathbb{T}$-models (with the notation $N^f_{\mathbb{T}}$ (risp. $M^g_{\mathbb{T}}$), we mean the $\mathbb{T}$-model which corresponds to the geometric morphism $f \circ n$ (resp. $g \circ m$)).

A theory classified by a comma topos

Proposition

Let

$$\begin{array}{ccc} (f/g) & \xrightarrow{\pi''} & \mathcal{E}_{\mathbb{T}''} \\ {\scriptstyle \pi'}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle f} \\ \mathcal{E}_{\mathbb{T}'} & \xrightarrow[g]{} & \mathcal{E}_{\mathbb{T}} \end{array}$$

be the comma topos of a pair of geometric morphisms $f, g \colon \mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$. *We assume that* f *(resp.* g*) is induced by some interpretation* $p \colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ *(resp.* $q \colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}''}$*).*
We consider

- *the signature* Σ''' *which is obtained by adding to* $\Sigma' \coprod \Sigma''$ *a relation symbol* $R^A \to A^{1p}, \ldots, A^{np}, A^{1q}, \ldots, A^{mq}$ *for every sort* A *in* Σ*, where the sorts* $A^{1p}, \ldots, A^{np}$ *(resp.* $A^{1q}, \ldots, A^{mq}$*) are those which arise in the context of* $p(\{x^A.\top\}) = \{\underline{x}^{A^p}.\psi^A\}$ *(resp.* $q(\{x^A.\top\}) = \{\underline{x}^{A^q}.\varphi^A\}$*);*
- *the theory* $\mathbb{T}'''$ *built over* Σ''' *which is obtained by adding to* $\mathbb{T}' \coprod \mathbb{T}''$ *the axioms:*

A theory classified by a comma topos

Proposition

1. *for every sort A in Σ,*

$$\psi^A \vdash_{\underline{x}^{A^p}} (\exists \underline{x}^{A^q}) R^A,$$

$$R^A \vdash_{\underline{x}^{A^p}, \underline{x}^{A^q}} \psi^A \wedge \varphi^A,$$

$$R^A \wedge R(\underline{x}^{A^q} | \underline{x'}^{A^q}) \vdash_{\underline{x}^{A^p} \underline{x}^{A^q}, \underline{x'}^{A^q}} \underline{x}^{A^q} = \underline{x'}^{A^q};$$

2. *for every function symbol $f \colon A^1, \ldots, A^n \to B$ in Σ,*

$$\top \vdash_{\underline{x}^{A^{1p}}, \ldots, \underline{x}^{A^{np}}, \underline{x}^{B^q}} ((\exists \underline{x}^{B^p}) \theta_f^p \wedge R^B) =$$

$$((\exists \underline{x}^{A^{1q}}, \ldots, \underline{x}^{A^{nq}}) R^{A^1} \wedge \cdots \wedge R^{A^n} \wedge \theta_f^q),$$

where θ_f^p (resp. θ_f^q) is a representative of the arrow $p([f(x^{A^1}, \ldots x^{A^n}) = y^B])$ (resp. $q([f(x^{A^1}, \ldots x^{A^n}) = y^B])$;

A theory classified by a comma topos

Proposition

3. *for every relation symbol* $R \to A^1, \dots A^n$ *in* Σ,

$$((\exists \underline{x}^{A^{1p}}, \dots, \underline{x}^{A^{np}})\theta_R^p \wedge R^{A^1} \wedge \cdots \wedge R^{A^n}) \vdash_{\underline{x}^{A^{1p}}, \dots, \underline{x}^{A^{np}}, \underline{x}^{A^{1q}}, \dots, \underline{x}^{A^{nq}}} \theta_R^q,$$

where θ_R^q *(resp.* θ_R^p*) is a formula which presents the subobject* $q([R(x^{A^1}, \dots x^{A^n})])$ *(resp.* $p([R(x^{A^1}, \dots x^{A^n})])$*).*

Then $\mathbb{T}'''$ *is a theory classified by the topos* (f/g).

A theory classified by a fibred product of toposes

- Let the diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathbb{T}'} \times_{\mathcal{E}_{\mathbb{T}}} \mathcal{E}_{\mathbb{T}''} & \xrightarrow{\pi''} & \mathcal{E}_{\mathbb{T}''} \\ {\scriptstyle \pi'}\downarrow & \Downarrow \varphi & \downarrow{\scriptstyle f} \\ \mathcal{E}_{\mathbb{T}'} & \xrightarrow[g]{} & \mathcal{E}_{\mathbb{T}} \end{array}$$

  be a pullback in **GrTop**, where f (risp. g) is induced by some interpretation $p\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ (risp. $q\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}''}$).
- We can describe a theory $\mathbb{T}''''$ classified by the fibred product $\mathcal{E}_{\mathbb{T}'} \times_{\mathcal{E}_{\mathbb{T}}} \mathcal{E}_{\mathbb{T}''}$ as a quotient of the theory $\mathbb{T}'''$ classified by the comma topos (f/g) described in the previous proposition. Indeed, by the universal property of our fibred product we deduce that a $\mathbb{T}''''$-model H in a topos $\mathcal{H}$ is, up to isomorphisms, a triplet (M, N, η), where M is a $\mathbb{T}'$-model in $\mathcal{H}$, N is a $\mathbb{T}''$-model in $\mathcal{H}$ and $\eta\colon N^f_{\mathbb{T}} \to M^g_{\mathbb{T}}$ is an isomorphism of $\mathbb{T}$-models.
- By what we have just said, we can describe a theory classified by the topos $\mathcal{E}_{\mathbb{T}'} \times_{\mathcal{E}_{\mathbb{T}}} \mathcal{E}_{\mathbb{T}''}$ by adding to the theory $\mathbb{T}'''$ of the previous proposition the following axioms:

A theory classified by a fibred product of toposes

1. for every sort A in Σ,

$$\varphi^A \vdash_{\underline{x}^{A^q}} (\exists \underline{x}^{A^p}) R_A(\underline{x}^{A^p}, \underline{x}^{A^q})$$

and

$$R_A(\underline{x}^{A^p}, \underline{x}^{A^q}) \wedge R_A(\underline{x'}^{A^p}, \underline{x}^{A^q}) \vdash_{\underline{x}^{A^p} \underline{x'}^{A^p}, \underline{x}^{A^q}} \underline{x}^{A^p} = \underline{x'}^{A^p},$$

2. for every relation symbol $R \to A^1, \ldots, A^n$ in Σ,

$$((\exists (\underline{x}^{A^{1p}}, \ldots, \underline{x}^{A^{np}}) \theta_R^q \wedge R_{A^1} \wedge \cdots \wedge R_{A^n})) \vdash_{\underline{x}^{A^{1p}}, \ldots, \underline{x}^{A^{np}}, \underline{x}^{A^{1q}}, \ldots, \underline{x}^{A^{nq}}} \theta_R^p,$$

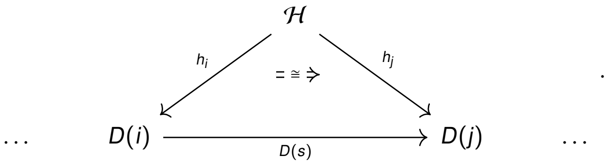
where θ_R^p is a formula which presents the subobject $p([R(x^{A^1}, \ldots x^{A^n})])$.

Limit of toposes: from the universal property to the semantics

- Let $D\colon \mathcal{I} \to \mathbf{GrTop}$ be a small diagram such that every topos $D(i)$ is a classifying topos of a theory $\mathbb{T}_i$. The universal property of the limit $\mathcal{L} := Lim_i D$ can be expressed as follows: for every topos $\mathcal{H}$ we have an equivalence of categories

$$\mathbf{GrTop}(\mathcal{H}, \mathcal{L}) \simeq \mathbf{Cone}(\mathcal{H}, (D)).$$

- If $\mathbb{T}$ is a theory classified by $\mathcal{L}$, then a $\mathbb{T}$-model H in a topos $\mathcal{H}$ can be regarded as a cone of D, i.e. a diagram in **GrTop** of the form



- By the universal property of classifying toposes we can regard such a cone as a collection $\{H^i\}_{i \in ob(\mathcal{I})}$, where H^i is a $\mathbb{T}_i$-model in $\mathcal{H}$, such that for every arrow $s\colon i \to j$ in $\mathcal{I}$ there exists an isomorphism $\overline{\alpha_s}\colon H^i_{\mathbb{T}_j} \to H^j$ of $\mathbb{T}_j$-models in $\mathcal{H}$, where $H^i_{\mathbb{T}_j}$ is the model that corresponds to $D(s) \circ h_i$.

A theory classified by a limit of toposes

Proposition

*Let $D \colon \mathcal{I} \to$ **GrTop** be a small diagram such that for every object i in $\mathcal{I}$ we have $D(i) = \mathcal{E}_{\mathbb{T}_i}$ and for every arrow $s \colon i \to j$ in $\mathcal{J}$ the geometric morphism $D(s) \colon \mathcal{E}_{\mathbb{T}_i} \to \mathcal{E}_{\mathbb{T}_j}$ is induced by an interpretation $p_s \colon \mathcal{C}_{\mathbb{T}_j} \to \mathcal{C}_{\mathbb{T}_j}$.*
We consider

- *the signature Σ that is obtained by adding for every arrow $s \colon i \to j$ in $\mathcal{I}$ and every sort A in Σ_j a relation symbol $R^s_A \to {A_1}'_i, \ldots, {A_n}'_i, A_j$ to $\coprod_i \Sigma_i$, where ${A_1}', \ldots, {A_n}'$ are sorts which arise in the context of the object $p_s(\{x^A.\top\}) = \{\underline{x}^{A'}.\psi^A\}$ in $\mathcal{C}_{\mathbb{T}_i}$, and for every object i in $\mathcal{I}$ the notation B_i denotes the embedding in $\coprod_i \Sigma_i$ of some sort B in Σ_i;*
- *the theory $\mathbb{T}$ built on the signature Σ that is obtained by adding to $\coprod_i \mathbb{T}_i$ the axioms*

A theory classified by a limit of toposes

Proposition

1 *for every arrow $s\colon i \to j$ in $\mathcal{I}$ and every sort A in Σ_j,*

$$\psi_i^A \vdash_{\underline{x}^{A_i'}} (\exists x^{A_j}) R_A^s,$$

$$R_A^s \vdash_{\underline{x}^{A_i'}, x^{A_j}} \psi_i^A,$$

$$R_A^s \wedge R_A^s[x^{A_i}|x'^{A_i}] \vdash_{\underline{x}^{A_i'}, x^{A_j}, x'^{A_i}} x^{A_i} = x'^{A_i},$$

$$\top \vdash_{x^{A_j}} (\exists \underline{x}^{A_i'}) R_A^s,$$

$$R_A^s(\underline{x}^{A_i'}, x^{A_j}) \wedge R_A^s(\underline{x'}^{A_i'}, x^{A_j}) \vdash_{\underline{x}^{A_i'}, \underline{x'}^{A_i'}, x^{A_j}} \underline{x}^{A_i'} = \underline{x'}^{A_i'},$$

where $\{\underline{x}^{A_i'}.\psi_i^A\}$ denotes the embedding of $\{\underline{x}^{A'}.\psi^A\}$ in $\mathcal{C}_{\coprod_i \mathbb{T}_i}$,

Proposition

2 *for every arrow $s : i \to j$ in $\mathcal{I}$ and for every function symbol $z : A_1, \dots A_n \to B$ in Σ_j,*

$$\top \vdash_{\underline{x}^{A_1}{}'_i, \dots, \underline{x}^{A_n}{}'_i, x^{B_i}} ((\exists \underline{x}^{B_i}{}')\theta_z \wedge R^s_B) = ((\exists x^{A_1 j}, \dots, x^{A_n j})R^s_{A_1} \wedge \cdots \wedge R^s_{A_n} \wedge$$

$$z_j(x^{A_1 j}, \dots, x^{A_n j}) = x^{B_i}),$$

3 *for every arrow $s : i \to j$ in $\mathcal{I}$ and for every relation symbol $R \to A_1, \dots, A_n$, in Σ_j,*

$$((\exists \underline{x}^{A_1}{}'_i, \dots, \underline{x}^{A_n}{}'_i)\theta_R \wedge R^s_{A_1} \wedge \cdots \wedge R^s_{A_n}) \vdash_{\underline{x}^{A_1}{}'_i, \dots, \underline{x}^{A_n}{}'_i, x^{A_1 j}, \dots, x^{A_n j}} R_j(x^{A_1 j}, \dots, x^{A_n j})$$

and

$$((\exists x^{A_1 j}, \dots, x^{A_n j})R_j(x^{A_1 j}, \dots, x^{A_n j}) \wedge R^s_{A_1} \wedge \cdots \wedge R^s_{A_n}) \vdash_{\underline{x}^{A_1}{}'_i, \dots, \underline{x}^{A_n}{}'_i, x^{A_1 j}, \dots, x^{A_n j}} \theta_R,$$

where z_j (resp. R_j) denotes the embedding of z (resp. R) in $\coprod_i \Sigma_i$, θ_z is a representative of $p_s([z(x^{A_1}, \dots, x^{A_1}) = y^B])$ and θ_R is a formula which presents the subobject $p_s(R(x^{A_1}, \dots, x^{A_1}))$.

Then $\mathbb{T}$ is a theory classified by the topos $\mathcal{L} := \lim_i D$.

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