Toposes with dependent and codependent arrows

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Toposes in Mondovì
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Dependent functions in MLTT and BST are fundamental objects

$$A : \mathcal{U}$$

$$P : A \to \mathcal{U}$$

$$\Phi : \prod_{x : A} P(x)$$

$$\Phi(a) : P(a), \quad a : A$$

If P is a constant type family i.e., $P(x) :\equiv B$, for every $x : A$, where $B : \mathcal{U}$, then $\Phi : \prod_{x : A} B$ is a term of the function type $A \to B$.

In MLTT (and in BST) the $\prod$-type is independent from the $\sum$-type

$$A : \mathcal{U}$$

$$P : A \to \mathcal{U}$$

$$(a, u) : \sum_{x : A} P(x),$$

$$a : A \quad \& \quad u \in P(a)$$

If P is a constant type family i.e., $P(x) := B$, for every $x : A$, where $B : \mathcal{U}$, then $(a, u) : \sum_{x : A} B$ is a term of the product type $A \times B$.

$$\mathtt{pr}_1 : \left(\sum_{x : A} P(x) \right) \to A, \quad (a, u) \mapsto a,$$

$$\mathtt{pr}_2 : \prod_{z : \sum_{x : A} P(x)} P(\mathtt{pr}_1(z)), \quad \mathtt{pr}_2\big((a, u)\big) := u.$$

In the categorical interpretation [27] of dependent type theory of Pitts the $\prod$-type is not fundamental and it depends on the corresponding categorical implementation of the $\sum$-type.

Our Starting Main Question: what is the categorical analogue to dependent functions?

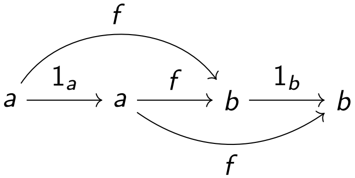
This question is different from finding categorical models for the whole of MLTT.

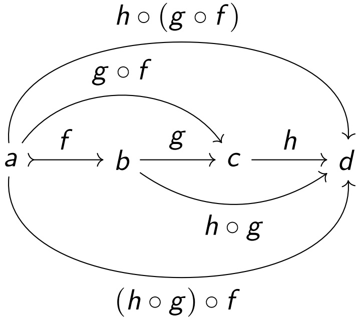
We want to model the $\prod$-type categorically

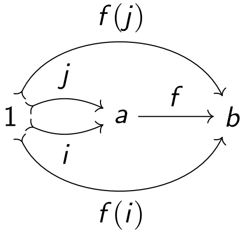
- as a fundamental notion,
- independent from a corresponding implementation of the $\sum$-type,
- and without requiring a strong background on MLTT.

How arrows generalise functions

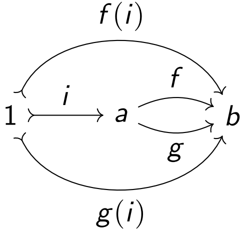
They preserve some properties of functions



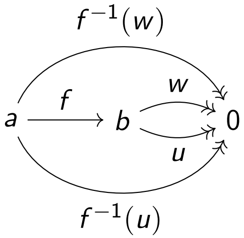


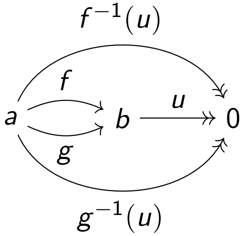


They forget some properties of functions

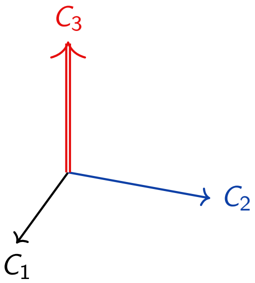


They add some properties that cannot be traced to functions





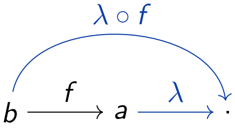
To C_1 we add family-arrows C_2 and dependent arrows C_3



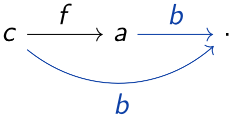
Dependent Category Theory

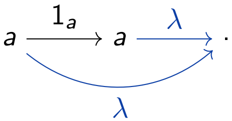
Categories with family-arrows $\lambda \in \mathtt{fHom}(a)$

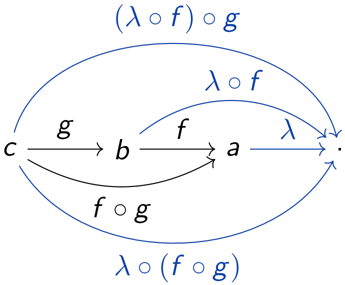
$$a \xrightarrow{\lambda} \cdot$$



$$a \xrightarrow{b} \cdot$$




1_a
a
a
λ
λ

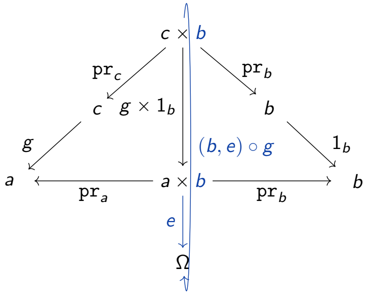

$(\lambda \circ f) \circ g$
$\lambda \circ f$
c
g
b
f
a
λ
$f \circ g$
$\lambda \circ (f \circ g)$

Family-arrows on a topos $\mathcal{C}$ (Pitts)

$$\mathtt{fHom}(a) := \bigcup_{b \in C_0} \mathrm{Hom}(a \times b, \Omega)$$

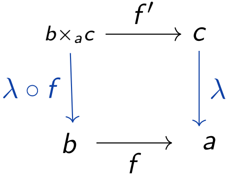
If $g : c \to a$, then

$$(b, e) \circ g := \big(b, e \circ (g \times 1_b)\big)$$



Weak family-arrows on a category with pullbacks (Seely)

If $\texttt{wfHom}(a) := \mathcal{C}/a$, then $(\texttt{fam}_1, \texttt{fam}_2)$ hold only up to isomorphism



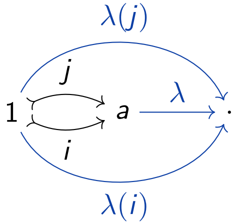
Debate over strict conditions vs weak conditions

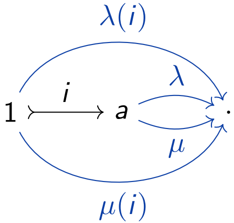
Bénabou, Ehrhard: Weak concepts are more general and more categorical, since isomorphism is a “more categorical” concept than equality.

Pitts: the strict approach suits better to modeling dependent type theory

Family-arrows preserve and forget properties of families of types

If $\mathcal{C}$ has 1 and $\lambda \in \texttt{fHom}(a)$, then $i = j \Rightarrow \lambda(i) = \lambda(j)$.





$\mathcal{C}$ with 1 has the *family-arrow-extensionality property* (farExt), if

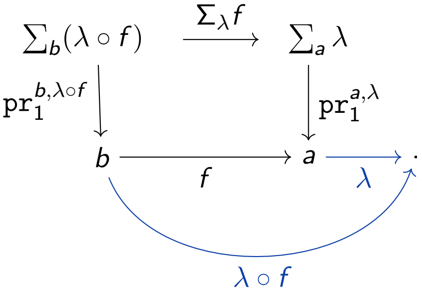
$$\forall_{i \in a}(\lambda(i) = \mu(i)) \Rightarrow \lambda = \mu$$

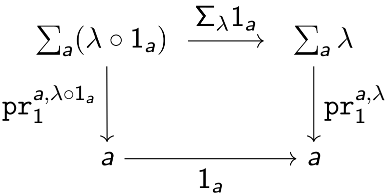
If $\texttt{fHom}(a) := a/\mathcal{C}$, the coslice of $\mathcal{C}$ over a, and composition $\lambda \circ f$ the composition in $\mathcal{C}$, then $\mathcal{C}$ has (farExt) if and only if $\mathcal{C}$ has (arExt).

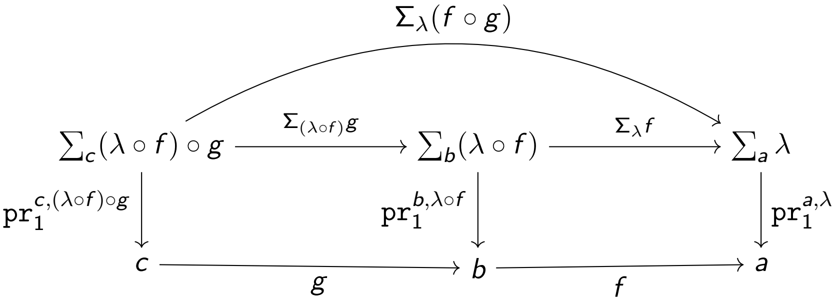
Categories with family-arrows and Sigma-objects

$$\Sigma_{\mathcal{C}} := \Bigg(\sum_a \lambda \in C_0,\ \mathtt{pr}_1^{a,\lambda} \colon \sum_a \lambda \to a \in C_1,$$

$$\Sigma_\lambda f \colon \sum_b (\lambda \circ f) \to \sum_a \lambda \in C_1 \Bigg)_{a,b \in C_0, \lambda \in \mathtt{fHom}(a), f \in \mathrm{Hom}(b,a)}$$







$(\mathtt{fam}, \Sigma)$-categories with 1 are the type-categories of Pitts (or Cartmell's categories with attributes).

If $(R, +, 0, \cdot, 1)$ is a commutative ring, and if $\mathcal{C}(R, +, 0)$ is the category of its additive, group-structure with objects a singleton $\{*\}$ and arrows the elements of R, then every commutative square

$$\begin{array}{ccc} * & \xrightarrow{\;a\;} & * \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ * & \xrightarrow[\;c\;]{} & * \end{array}$$

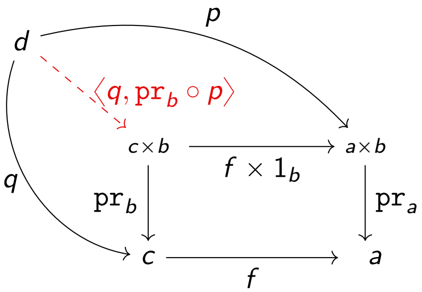
is a pullback. Let $\mathtt{Fam}(*) := R \times R$ and $(a, b) \circ c := (c + a, c + b)$. Let $\sum_*(a, b) := *$, $\mathtt{pr}_1^{*,(a,b)} := a \cdot b$, $\Sigma_{(a,b)} c := c(1 + c + b + a)$,

$$\begin{array}{ccc} * & \xrightarrow{\;c(1 + c + b + a)\;} & * \\ {\scriptstyle (c + a) \cdot (c + b)}\big\downarrow & & \big\downarrow{\scriptstyle a \cdot b} \\ * & \xrightarrow[\;c\;]{} & * \end{array}$$

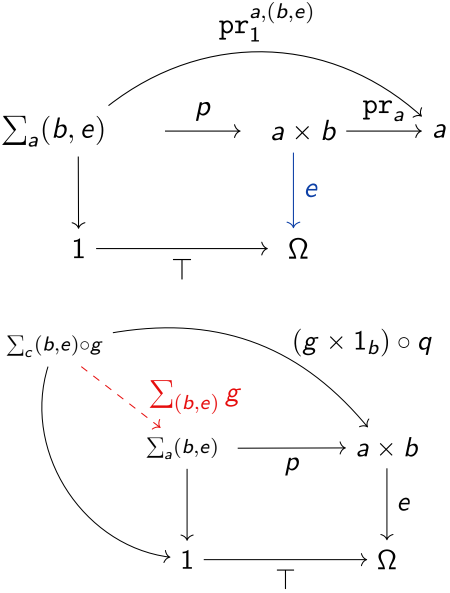
$\mathcal{C}(R, +, 0)$ is a $(\mathtt{fam}, \Sigma)$-category, which, in general, has no 1.

If $\mathcal{C}$ has binary products and $b \in \mathtt{fHom}(a)$,

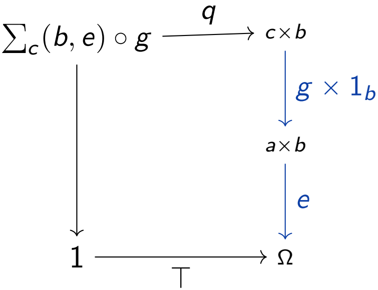
$$\sum_a b := a \times b \ \ \& \ \ \mathtt{pr}_1^{a,b} := \mathtt{pr}_a \colon a \times b \to a.$$



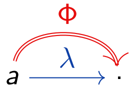
Sigma-objects on a topos I (Pitts)

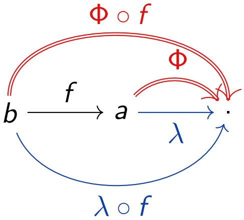


Sigma-objects on a topos II (Pitts)



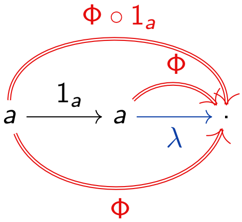
Categories with dep-arrows $\Phi \in \mathtt{dHom}(a, \lambda), \lambda \in \mathtt{fHom}(a)$

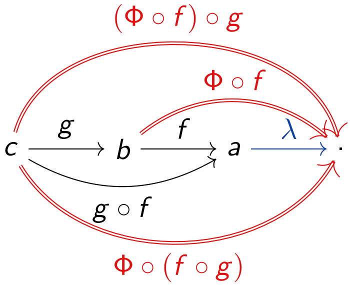


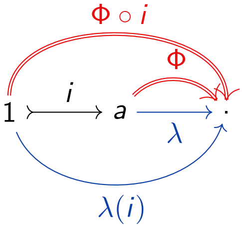


$$f = g \Rightarrow \Phi \circ f = \Phi \circ g$$

$$\Phi = \Psi \Rightarrow \Phi \circ f = \Psi \circ f$$


Φ ∘ 1_a
Φ
1_a
a
a
λ
Φ


(Φ ∘ f) ∘ g
Φ ∘ f
g
f
λ
c
b
a
g ∘ f
Φ ∘ (f ∘ g)

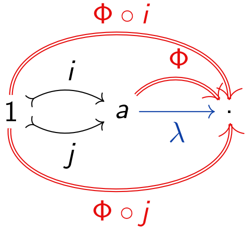


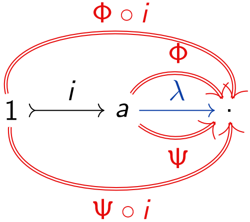
If i is a global element of a, then $\Phi \circ i$, or $\Phi(i)$, is in $\mathtt{dHom}(1, \lambda(i))$,

$$\prod_{j\,:\,1} P(j)$$

can be identified to $P(0_1)$.

Dep-arrows preserve and forget properties of dependent functions





A dep-category $\mathcal{C}$ with 1 has the dependent-arrow-extensionality property (darExt), if $\forall_{i \in a}(\Phi(i) = \Psi(i)) \Rightarrow \Phi = \Psi$

Any category $\mathcal{C}$ is turned into a dep-category

$$\mathtt{fHom}(a) := C_0$$

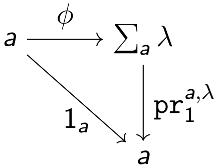
$$\mathtt{dHom}(a, b) := \mathrm{Hom}(a, b)$$

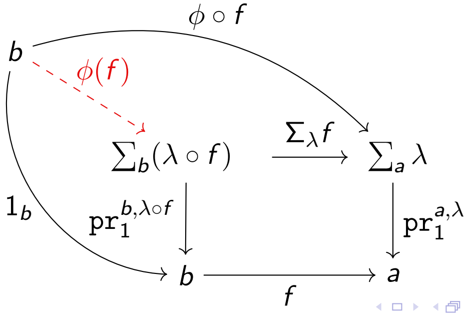
$$f \circ g \in \mathtt{dHom}(c, b \circ g) := \mathtt{dHom}(c, b) := \mathrm{Hom}(c, b),$$

for every $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(c, a)$.

Any $(\mathtt{fam}, \Sigma)$-category is turned into a dep-category

$$\mathtt{dHom}(a, \lambda) := \mathcal{D}_a\lambda := \left\{ \phi \in \mathrm{Hom}\left(a, \sum_a \lambda\right) \mid \mathtt{pr}_1^{a,\lambda} \circ \phi = 1_a \right\}$$





The canonical dep-structure on a topos $\mathcal{C}$

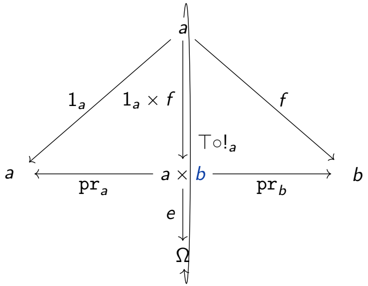
Theorem (Ehrhardt)

If $a \in \mathcal{C}$ *and* $(b, e) \in \mathtt{fHom}(a)$ *i.e.,* $e\colon a \times b \to \Omega$*, then*

$$\mathtt{dHom}(a, (b, e)) := \left\{ \phi\colon\ \in \mathrm{Hom}\left(a, \sum_a (b, e)\right) \mid \mathtt{pr}_1^{(a,(b,e))} \circ \phi = 1_a \right\}$$

is bijective to

$$\{ f \in \mathrm{Hom}(a, b) \mid e \circ \langle 1_a, f \rangle = \top \circ !_a \}$$



There are dep-structures that are not induced by the corresponding $(\mathtt{fam}, \Sigma)$-structures

The canonical dep-structure on a commutative ring is the singleton

$$\mathtt{dHom}(*, (a, b)) := \{r \in R \mid ab + r = 0\},$$

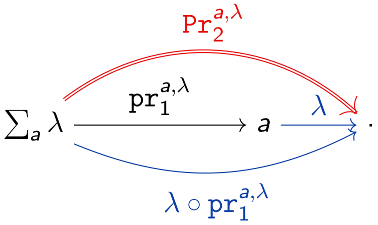
while one can define the following dep-structure

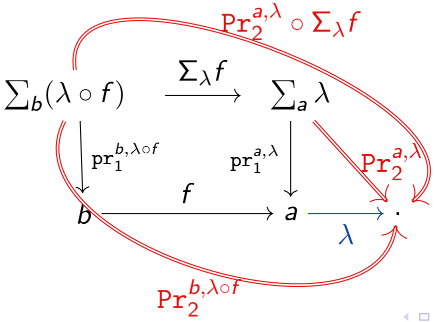
$$\mathtt{dHom}'(*, (a, b)) := \{I \in \mathtt{Ideal}(R) \mid a - b \in I\},$$

$$I \circ r := I, \quad r \in \mathrm{Hom}(*, *).$$

We can find trivially R and $a, b \in R$ with many ideals containing $a - b$.

Categories with dependent arrows and Sigma-objects

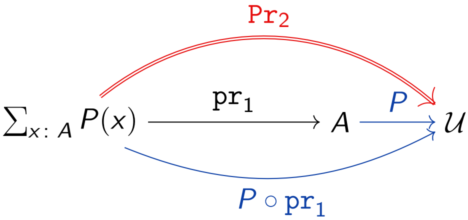




$$\mathtt{pr}_1 \colon \left(\sum_{x \colon A} P(x)\right) \to A, \quad \mathtt{pr}_1(a, b) := a$$

$$\mathtt{Pr}_2 \colon \prod_{z \colon \sum_{x \colon A} P(x)} P(\mathtt{pr}_1(z)), \quad \mathtt{Pr}_2(a, b) := b$$

$$z = \big(\mathtt{pr}_1(z), \mathtt{Pr}_2(z)\big)$$



If $\mathcal{C}$ has binary products and $b \in \mathtt{fHom}(a)$,

$\mathcal{C}$ is turned into a $(\mathtt{dep}, \Sigma)$-category:

$\mathtt{Pr}_2^{a,b} := \mathtt{pr}_b \in \mathtt{dHom}(a \times b, b \circ \mathtt{pr}_a) := \mathtt{dHom}(a \times b, b) := \mathrm{Hom}(a \times b, b)$,

and by the definition of $f \times 1_b$ we get

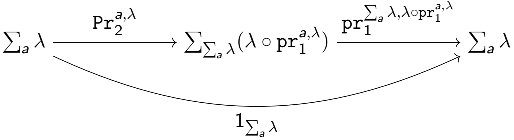
$$\mathtt{Pr}_2^{a,b} \circ \Sigma_b f := \mathtt{pr}_b \circ (f \times 1_b) = \mathtt{pr}_b =: \mathtt{Pr}_2^{c,b} = \mathtt{Pr}_2^{c,b \circ f}.$$

Theorem

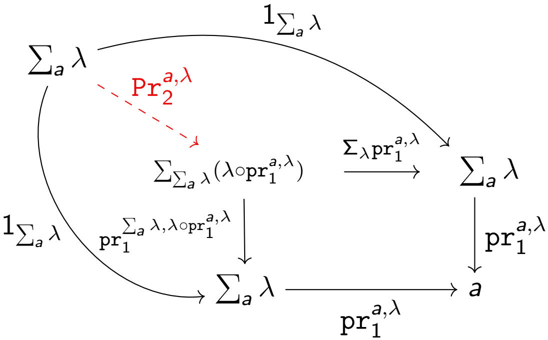
A $(\mathtt{fam}, \Sigma)$*-category* $\mathcal{C}$ *is turned into a* $(\mathtt{dep}, \Sigma)$*-category:*

$$\mathtt{Pr}_2^{a,\lambda} \in \mathcal{D}_{\sum_a \lambda}\left(\lambda \circ \mathtt{pr}_1^{a,\lambda}\right) =$$

$$\left\{\phi \in \mathrm{Hom}\left(\sum_a \lambda, \sum_{\sum_a \lambda} (\lambda \circ \mathtt{pr}_1^{a,\lambda})\right) \mid \mathtt{pr}_1^{\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}} \circ \mathtt{Pr}_2^{a,\lambda} = 1_{\sum_a \lambda}\right\}$$



Proof.


$\sum_a \lambda$
$1_{\sum_a \lambda}$
$\mathtt{Pr}_2^{a,\lambda}$
$\sum_{\sum_a \lambda}(\lambda \circ \mathtt{pr}_1^{a,\lambda})$
$\Sigma_\lambda \mathtt{pr}_1^{a,\lambda}$
$\sum_a \lambda$
$1_{\sum_a \lambda}$
$\mathtt{pr}_1^{\sum_a \lambda, \lambda \circ \mathtt{pr}_1^{a,\lambda}}$
$\mathtt{pr}_1^{a,\lambda}$
$\sum_a \lambda$
a
$\mathtt{pr}_1^{a,\lambda}$

□

There are $(\mathtt{dep}, \Sigma)$-structures that are not induced by the corresponding $(\mathtt{fam}, \Sigma)$-structures

Let non-canonical dep-structure on a commutative ring

$$\mathtt{dHom}'(*, (a, b)) := \{I \in \mathtt{Ideal}(R) \mid a - b \in I\},$$

$$I \circ r := I, \quad r \in \mathrm{Hom}(*, *).$$

We can define

$$\mathtt{Pr}_2^{*,(a,b)} := \langle a - b \rangle \in \mathtt{dHom}'(*, (a, b) \circ ab) :=$$

$$\mathtt{dHom}'(*, (ab + a, ab + b)) = \mathtt{dHom}'(*, (a, b)).$$

A generalisation of the Grothendieck construction

Theorem

Let $\mathcal{C}$ be a $(\mathtt{fam}, \Sigma)$-category, and let $\Sigma(\mathcal{C})$ be the category of Sigma-objects of $\mathcal{C}$ with arrows $(f, \Sigma f)\colon \sum_b \mu \to \sum_a \lambda$

$$\begin{array}{ccc} \sum_b \mu & \xrightarrow{\Sigma f} & \sum_a \lambda \\ {\scriptstyle \mathtt{pr}_1^{b,\mu}} \big\downarrow & & \big\downarrow {\scriptstyle \mathtt{pr}_1^{a,\lambda}} \\ b & \xrightarrow[f]{} & a. \end{array}$$

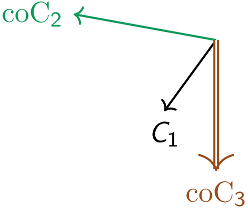
Then $\mathtt{pr}_1\colon \Sigma(\mathcal{C}) \to \mathcal{C}$, where $\sum_a \lambda \mapsto a$ and $(f, \Sigma f) \mapsto f$, is a split fibration with

$$\left[\gamma\left(\sum_a \lambda\right)\right](f) := \left(f, \sum_\lambda f\right)$$

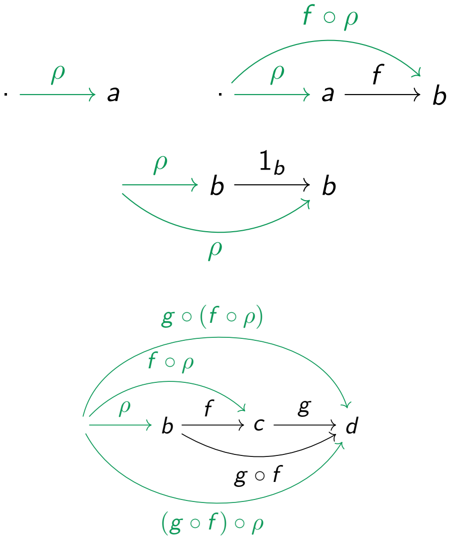
as splitting cleavage.

What we add that is not traced to dependent functions

To C_1 we add cofamily-arrows coC_2 and codependent arrows coC_3

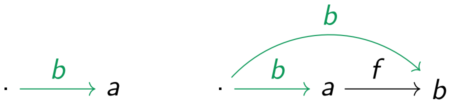


Categories with cofamily-arrows $\rho \in \texttt{cofHom}(a)$



Any category $\mathcal{C}$ is turned into a cofam-category

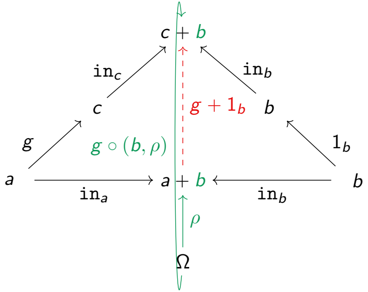
$$\texttt{cofHom}(a) := C_0$$



Cofamily-arrows on a topos $\mathcal{C}$

$$\mathtt{cofHom}(a) := \bigcup_{b \in C_0} \mathrm{Hom}(\Omega, a + b)$$

If $g\colon a \to c$, then $g \circ (b, \rho) := \big(b, (g + 1_b) \circ \rho\big)$.



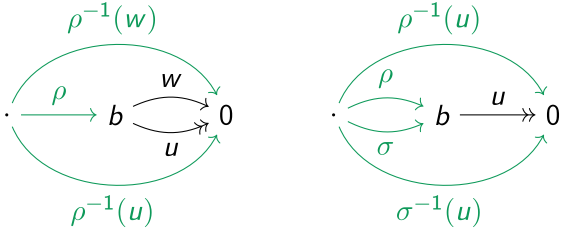
Weak cofamily-arrows on a category with pushouts

If $\texttt{wcofHom}(a) := a/\mathcal{C}$, then $(\texttt{cofam}_1, \texttt{cofam}_2)$ hold only up to isomorphism

$$\begin{array}{ccc} a & \xrightarrow{\ f\ } & b \\ \rho \downarrow & & \downarrow f \circ \rho \\ c & \xrightarrow[\ f'\ ]{} & c +_a b \end{array}$$

If a `cofam`-category $\mathcal{C}$ has 0, then $u = w \Rightarrow \rho^{-1}(u) = \rho^{-1}(w)$, and $\mathcal{C}$ has the cofamily-arrow-extensionality property $(\mathrm{cofarrExt})$, if

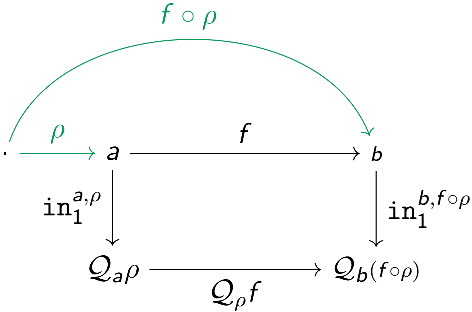
$$\forall_{u \in^{\mathrm{op}} b}(\rho^{-1}(u) = \sigma^{-1}(u)) \Rightarrow \rho = \sigma.$$



$\mathcal{C}$ becomes a `cofam`-category by taking the slices as cofamily-arrows, and then $\mathcal{C}$ has $(\mathrm{cofarrExt})$ if and only if $\mathcal{C}$ has $(\mathrm{arcoExt})$.

Categories with cofamily arrows and coSigma-objects

$$\mathcal{Q}_{\mathcal{C}} := \Big(\mathcal{Q}_a\rho \in C_0,\ \mathtt{in}_1^{a,\rho}\colon a \to \mathcal{Q}_a\lambda \in C_1,$$
$$\mathcal{Q}_\rho f\colon \mathcal{Q}_a\rho \to \mathcal{Q}_b(f \circ \rho) \in C_1\Big)_{a,b\in C_0,\rho\in\mathtt{cofHom}(a),f\in\mathrm{Hom}(a,b)}$$

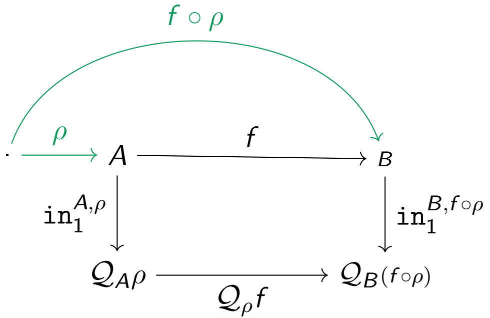


In **Set**, if $\rho\colon X \to A$ in $\mathtt{cofHom}(A)$, let

$$\mathcal{Q}_A\rho := \{\rho^{-1}(a) \mid a \in A\}$$

$$\mathtt{in}^1_{A,\rho}\colon A \to \mathcal{Q}_A\rho, \qquad a \mapsto \rho^{-1}(a)$$

$$[Q_\rho f](\rho^{-1}(a)) := (f \circ \rho)^{-1}(f(a))$$

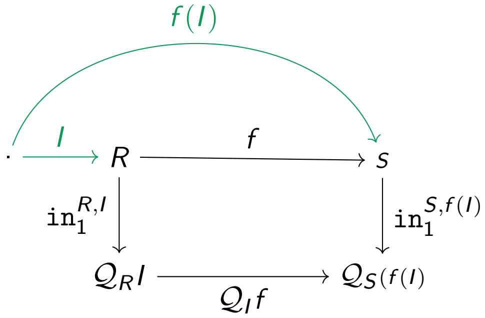


In **Ring** with arrows $f\colon R \to S$ ring-epimorphisms, if $\texttt{cofHom}(R) := \mathcal{I}(R)$, the ideals of R, with $f \circ I := f(I)$, then

$$\mathcal{Q}_R I := R/I$$

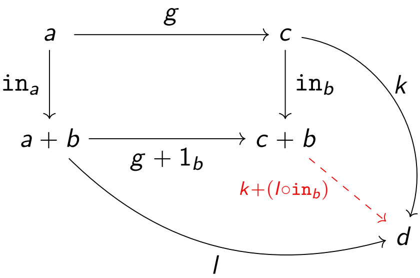
$$\texttt{in}^1_{A,\rho}\colon A \to \mathcal{Q}_A\rho, \qquad r \mapsto r + I$$

$$\mathcal{Q}_I f\colon R/I \to S/I, \qquad [\mathcal{Q}_I f](r+I) := f(r) + f(I)$$

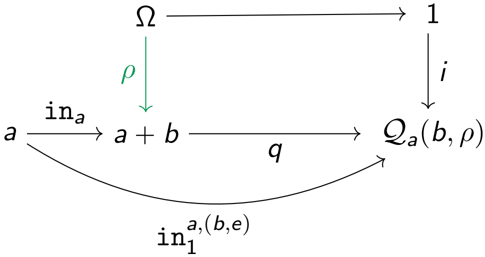


If $\mathcal{C}$ has binary coproducts and $b \in \mathtt{cofHom}(a)$,

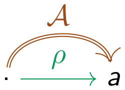
$$\mathcal{Q}_a b := a + b \;\;\&\;\; \mathtt{in}_1^{a,b} := \mathtt{in}_a \colon a \to a + b.$$

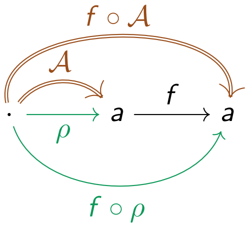


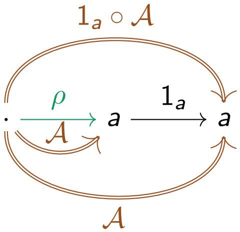
coSigma-objects on a topos

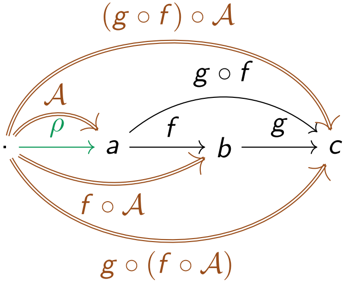


Cats with codep-arrows $\chi \in \texttt{codHom}(a, \rho), \rho \in \texttt{cofHom}(a)$









$$f = g \Rightarrow f \circ \mathcal{A} = g \circ \mathcal{A}$$

$$\mathcal{A} = \mathcal{B} \Rightarrow f \circ \mathcal{A} = f \circ \mathcal{B}$$

A $(\texttt{cofam}, \mathcal{Q})$-category is a codep-category

If $\mathcal{C}$ is a $(\texttt{cofam}, \mathcal{Q})$-category, let for every $a \in \mathcal{C}$ and $\rho \in \texttt{codHom}(a)$

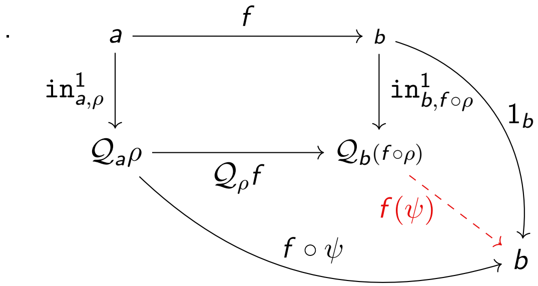
$$\mathcal{C}_a\rho := \{\psi \in \mathrm{Hom}(\mathcal{Q}_a\rho, a) \mid \psi \circ \texttt{in}^1_{a,\rho} = 1_a\}$$

$$a \xrightarrow{\texttt{in}^1_{a,\rho}} \mathcal{Q}_a\rho \xrightarrow{\psi} a, \qquad \psi \circ \texttt{in}^1_{a,\rho} = 1_a$$

be the codependent objects of ρ. If $\texttt{codHom}(\rho, a) := \mathcal{C}_a\rho$, then $\mathcal{C}$ becomes a codep-category.

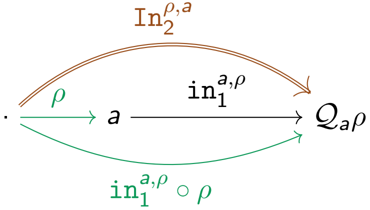
Proof:

If we write $f(\psi)$, instead of the used in the proof composition $f \circ \psi$, we get the required arrow by the universal property of pushouts.



Second injection $\mathtt{In}_2^{a,\rho}$

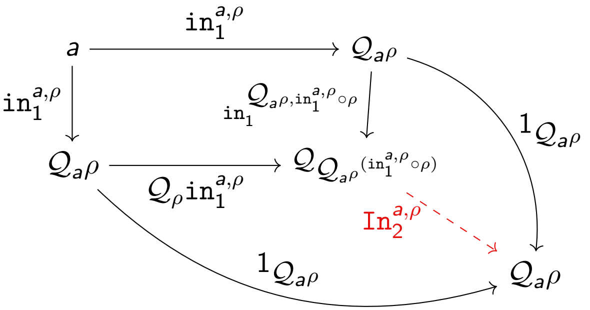
The dual to the dependent arrow $\mathtt{Pr}_2^{a,\lambda}$ is the codependent arrow $\mathtt{In}_2^{a,\rho} \in \mathtt{codHom}\left(\mathtt{in}_1^{a,\rho} \circ \rho, \mathcal{Q}_a \rho\right)$



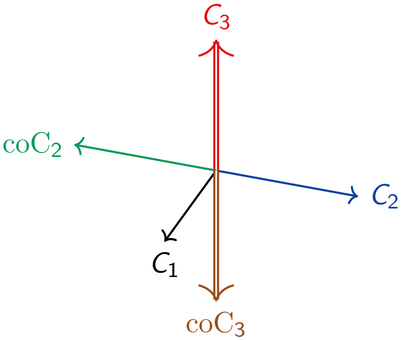
such that, for every $b \in \mathcal{C}$ and $f \in \mathrm{Hom}(a, b)$ we have that

$$\mathtt{In}_2^{b,f\circ\rho} = \mathcal{Q}_\rho f \circ \mathtt{In}_2^{a,\rho}.$$

A $(\mathtt{cofam}, \mathcal{Q})$-category is a $(\mathtt{codep}, \mathcal{Q})$-category in a canonical way



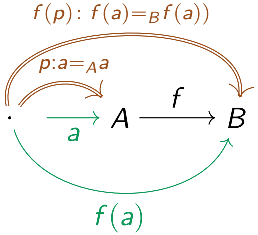
To C_1 we add C_2, C_3 and $\mathrm{coC}_2, \mathrm{coC}_3$



The category of small types $\mathcal{U}$

$$A\colon \mathcal{U}, \quad \mathtt{cofam}(A) := A, \quad \mathtt{codHom}(A, a) := \Omega(A, a) := a =_A a,$$

$$f \circ p := f(p)\colon \mathtt{codHom}(B, f(a)) := \Omega(B, f(a)) := f(a) =_B f(a).$$

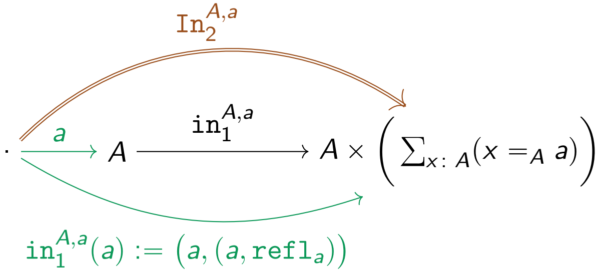


This codep-structure on $\mathcal{U}$ is not induced by the following $\mathcal{Q}$-structure on $\mathcal{U}$.

$$\mathcal{Q}_A a := A \times \left(\sum_{x \colon A} (x =_A a) \right),$$

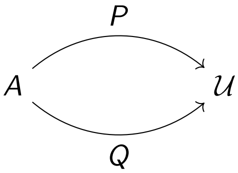
$$\mathtt{in}_1^{A,a} \colon A \to A \times \left(\sum_{x \colon A} (x =_A a) \right), \qquad a' \mapsto \big(a', (a, \mathtt{refl}_a)\big),$$

$$\mathtt{In}_2^{A,a} := \mathtt{refl}_{(a,(a,refl_a))} \colon \mathtt{codHom}\big(\mathtt{in}_1^{A,a} \circ a, \mathcal{Q}_A a\big).$$



The interplay between the dependent and codependent features of $\mathcal{U}$ is expected to lead to a good notion of type-category.

Small types form a 2-`fam`-category

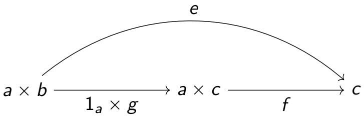


$$\mathrm{Hom}(P, Q) := \prod_{x\colon A} (P(x) \to Q(x))$$

A topos is a 2-$\mathtt{fam}$-category

If (b, e) and (c, f) are in $\mathtt{fHom}(a)$, then

$$\mathrm{Hom}\big((b, e), (c, f)\big) := \{g \in \mathrm{Hom}(b, c) \mid f \circ (1_a \times g) = e\}$$



Toposes are also 2-$(\mathtt{fam}, \Sigma)$-categories, 2-$\mathtt{dep}$-categories, and 2-$(\mathtt{dep}, \Sigma)$-categories (see [9]).

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