Sheaves on bicategories

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Introduction

- Bicategories are one of several definitions for 2-categories. They are a horizontal categorification of the notion of monoidal category, which is the basis for enrichment theory.
- The notion of enrichment over a bicategory has been known for some time [7] ; it generalizes the theory of enrichment over a monoidal category.
- On another hand, sheaves have been defined over quantaloids, which are a generalization of quantales, using a formalism of enrichment. As quantaloids are a particular case of bicategories, this suggests that the construction of sheaves over quantaloids can be extended to a construction of sheaves over a bicategory.

Sheaves on locales

- Recall that sheaves on a locale X can be expressed in several ways. Notably, we have the following description, of sheaves as "sets with a local equality" [6].

Definition

A sheaf on X is a set A together with an application $d : A \times A \to X$, satisfying :

- $d(x, y) = d(y, x)$
- $d(x, y) \wedge d(y, z) = d(x, z)$

- This description can be put in perspective by considering quantaloids, in particular the quantaloid of relations of X.

Sheaves on quantales

- Quantales are a non-commutative generalization of locales. They are complete lattice like locales, but with an additional operation representing a non-commutative intersection.
- This operation is the one we ask that is distributed through joins, not the usual meet.
- Any locale is a commutative quantale in which the additional operation coincides with the meet.
- As for locales it is possible to define sheaves on quantales as sets with an equality [2, 3, 10], but this time, more complicated conditions arise.
- This definition does give back the same category of sheaves over a locale considered as a quantale that what we would get by considering it as a locale.

Quantaloids

Definition

A quantaloid is a category for which all hom-sets are locally ordered in a way that respect composition : if we have two arrows $u, v : q_1 \to q_2$ then for any composable arrows $fu \leq fv$ and $ug \leq vg$. Moreover, we ask that composition respects joins : for any family $(u_i)_{i \in I} : q_1 \to q_2$ of arrows and any composable arrows : $f \bigvee_{i \in I} u_i = \bigvee_{i \in I} fu_i$ and $(\bigvee_{i \in I} u_i)g = \bigvee_{i \in I} u_i g$.

- In other words, quantaloids are **Sup**-enriched categories, where **Sup** is the category of complete lattices and join-preserving morphisms between them.
- Any quantale is precisely a one-object quantaloid.

Sheaves on quantaloids

- It is possible to define sheaves on quantaloids by generalizing the definition of sheaves on quantales and locales as sets with equality. This time, however, we will need to take a set that varies along the objects of the quantaloid [5, 9].
- The definition of sheaves over quantaloids uses a theory of enrichment over the base quantaloid $\mathcal{Q}$. This notion may not seem to be exactly the same as that of enrichment over a monoidal category, but it is a particular case of a greater theory of enrichment in bicategories, which also comprises that of enrichment over a monoidal category.

Definition

A sheaf over some quantaloid $\mathcal{Q}$ is a $\mathrm{Pr}(\mathcal{Q})$-category which is skeletal and Cauchy-complete.

Properties of $\mathbf{Sh}(\mathcal{Q})$

- Sheaves on quantaloids generalize sheaves on locales ; in fact, any locale X admits a quantaloid of relations of X, over which the quantaloid-theoretical sheaves are precisely sheaves over X [1].
- For any site $(\mathcal{C}, J)$ we can get a quantaloid $\mathcal{R}(\mathcal{C}, J)$ of closed relations such that $\mathbf{Sh}(\mathcal{R}(\mathcal{C}, J)) = \mathbf{Sh}(\mathcal{C}, J)$.
- In general, $\mathbf{Sh}(\mathcal{Q})$ is not a Grothendieck topos, because the lattices of subobject fail to satisfy the property of distributivity of colimits over pullbacks (they are not locales).

Bicategories : definition

- Bicategories are one of the several possible definitions for 2-categories. They are defined by asking that the composition functor is associative and unital up to isomorphism.
- More precisely, a bicategory $\mathcal{B}$ is given by a set of 0-cells or objects $\mathrm{Ob}(\mathcal{B})$, and for each par x, y of 0-cells, a hom-category $\mathcal{B}(x, y)$. The objects of $\mathcal{B}(x, y)$ are 1-cells while its arrows are 2-cells.
- On each 0-cell x, there is an identity 1-cell that we denote by id_x and on each 1-cell f, there is an identity 2-cell 1_f.
- A family of composition functors $c_{xyz} : \mathcal{B}(y, z) \times \mathcal{B}(x, y) \to \mathcal{B}(x, z)$ yields a composition of 1-cells, but also a horizontal composition of 2-cells, denoted by $f * g$.
- Several isomorphisms and coherence conditions dictate the behavior of a bicategory.

Bicategories : examples

- Any monoidal category can be expressed as a one-object bicategory, with the composition of 1-cells being given by the monoidal operation on the objects of the base monoidal category. The coherence conditions of the monoidal category coincide with those of the resulting bicategory.
- Quantaloids are a particular case of bicategories, in which the hom-categories are posets admitting coproducts, and such that the composition functor respects coproducts. Note that the coproduct in a quantaloid is an idempotent operation.
- In general, we call locally cocomplete a bicategory in which all the hom-categories admit those colimits being preserved by the composition functor.

Categorical operations in a bicategory

- Recall that a closed monoidal category is a monoidal category for which the monoidal operation has a right adjoint called the internal hom.
- In the corresponding one-object bicategory, this amounts to asking that the composition functor with one fixed variable has a right adjoint : the internal hom corresponds to right Kan extensions and right Kan lifts.
- In a quantaloid, the right Kan extension correspond to the right implication.
- If every hom-category admit right Kan extensions and lifts, we say that $\mathcal{B}$ is closed.
- We will work with $\mathcal{B}$ being closed and locally cocomplete ; that enable us in particular to compute coends, which are a particular case of colimit.

$\mathcal{B}$-categories

Definition

A $\mathcal{B}$-category is the data of :

- A pair $(\mathbb{M}, A)$, where A is a Ob$(\mathcal{B})$-typed set, i.e. a set together with a function $t : A \to \text{Ob}(\mathcal{B})$, and $\mathbb{M}$ is an endomatrix over A, i.e. for any $a, b \in A$, $\mathbb{M}(a, b) \in \mathcal{B}(tb, ta)$.
- For all $a, b, c \in A$, a idempotency 2-cell : $\iota_{abc} : \mathbb{M}(a, b)\mathbb{M}(b, c) \Rightarrow \mathbb{M}(a, c)$.
- For all $a \in A$, a reflexivity 2-cell : $\rho_a : \text{id}_{ta} \Rightarrow \mathbb{M}(a, a)$.

Satisfying some unitality and associativity conditions.

- This recovers the notion of enrichment in a monoidal category. Enriching in a monoidal category is the same as enriching in the corresponding bicategory.
- This also recovers the notion of enrichment in a quantaloid.

The category $\mathbf{Cat}(\mathcal{B})$

Definition

A $\mathcal{B}$-functor between two $\mathcal{B}$-categories $f : (\mathbb{M}, A) \to (\mathbb{N}, C)$ is a type-preserving function $f : A \to C$ together with a 2-cell for each $a, a' \in A$: $f_{aa'} : \mathbb{M}(a, a') \Rightarrow \mathbb{N}(f(a), f(a'))$, satisfying some coherence conditions.

- There is a category $\mathbf{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories and $\mathcal{B}$-functors between them.
- To define the notions of Cauchy-completion and skeletality on $\mathcal{B}$-categories, we need another kind of morphism between $\mathcal{B}$-categories : distributors.

Distributors between $\mathcal{B}$-categories

Definition

A distributor between two $\mathcal{B}$-categories $\phi : (\mathbb{M}, A) \to (\mathbb{N}, C)$ is the data of :

- A $\mathcal{B}$-matrix $\phi : A \to C$, i.e. $\phi(c, a) : ta \to tc$ for all $a \in A$, $c \in C$.
- A 2-cell $\delta_{cc'a'a} : \mathbb{N}(c, c')\phi(c', a')\mathbb{M}(a', a) \Rightarrow \phi(c, a)$ for all $a, a' \in A$, $c, c' \in C$.

Satisfying two sets of coherence conditions corresponding to the following bimodule conditions :

- $1 \cdot m \cdot 1 = m$
- $a \cdot (a' \cdot m \cdot b') \cdot b = (aa') \cdot m \cdot (b'b)$

The bicategory $\mathbf{Dist}(\mathcal{B})$

- Distributors can be composed one to another through the use of coends : for any two distributors $\psi : (\mathbb{M}, A) \to (\mathbb{N}, C)$ and $\phi : (\mathbb{N}, C) \to (\mathbb{P}, D)$, we have :

$$(\phi\psi)(d, a) = \int^{c:C} \phi(d, c)\psi(c, a)$$

- The matrix $\mathbb{M}$ is an identity distributor over any $\mathcal{B}$-category $(\mathbb{M}, A)$ for this composition.
- We thus get a category $\mathbf{Dist}(\mathcal{B})$ of $\mathcal{B}$-categories and distributors between them. It is possible to make it into a bicategory by considering the following 2-cells :

Definition

A morphism of distributors $\alpha : \psi \to \phi$, for $\psi, \phi : (\mathbb{M}, A) \to (\mathbb{N}, C)$, is the data for all $a \in A$, $c \in C$ of a 2-cell $\alpha_{a,c} : \psi(c, a) \to \phi(c, a)$, satisfying some coherence conditions.

Singletons of a $\mathcal{B}$-category

- To express completeness of a $\mathcal{B}$-category, we must define the following notion :

Definition

A singleton of $(\mathbb{M}, A)$ is a distributor $(\mathrm{id}_*, *) \to (\mathbb{M}, A)$, for some $* \in \mathrm{Ob}(\mathcal{B})$, which has a right adjoint.

- In this definition, $(\mathrm{id}_*, *)$ is the $\mathcal{B}$-category composed of the 1×1 matrix (id_*), with $* \in \mathrm{Ob}(\mathcal{B})$ over the $\mathrm{Ob}(\mathcal{B})$-typed set $\{*\}$ (of type $*$).
- The most important type of singleton is given by $\mathbb{M}(-, a)$ for some $a \in A$. Those singletons of that form are called the representable singleton.

Sheaves on a bicategory

Definition

A $\mathcal{B}$-category $(\mathbb{M}, A)$ is said to be :

- Skeletal if for any $a, b \in A$, $\mathbb{M}(-, a) = \mathbb{M}(-, b)$ implies $a = b$.
- Complete if any singleton of $(\mathbb{M}, A)$ is of the form $\mathbb{M}(-, a)$ for some $a \in A$.

- We denote by $\mathbf{Cat}_\kappa(\mathcal{B})$ the full subcategory of $\mathbf{Cat}(\mathcal{B})$ whose objects are skeletal and complete $\mathcal{B}$-categories.
- Recall that for a quantaloid $\mathcal{Q}$, $\mathbf{Sh}(\mathcal{Q})$ is defined as $\mathbf{Cat}_{\sigma\kappa}(\mathrm{Pr}(\mathcal{Q}))$. Sheaves on $\mathcal{B}$ should be defined as skeletal and complete categories enriched in a bicateory obtained from $\mathcal{B}$.
- In the case of the bicategory $\mathcal{B}_{\mathbf{Set}}$, skeletal complete $\mathcal{B}_{\mathbf{Set}}$-categories are usual Cauchy-complete categories.

Properties of $\mathbf{Cat}_\kappa(\mathcal{B})$

- In general, the category $\mathbf{Cat}_\kappa(\mathcal{B})$ is not a topos, because it contains the case of quantaloids.
- Also because of that, any Grothendieck topos can be recovered as some $\mathbf{Cat}_\kappa(\mathcal{B})$ (by taking the corresponding quantaloid for example).
- We are still investigating all properties of the category $\mathbf{Cat}_\kappa(\mathcal{B})$, but we are going to present an unfinished plan for a construction which exhibits it as a left-exact reflective subcategory of the category $[\mathbf{Map}(\mathcal{B})^{\mathsf{op}}, \mathbf{Cat}]$ of indexed categories over $\mathbf{Map}(\mathcal{B})$.

Sheaves are presheaves

Definition

Let $(\mathbb{M}, A)$ be a skeletal complete $\mathcal{B}$-category. Then define $P_{\mathbb{M},A}$ as follows :

- For any object x of $\mathcal{B}$, $\mathrm{Ob}(P_{\mathbb{M},A}(x))$ is the set of all elements of A of type x. We denote it by A_x.
- For any a, b of type x, an arrow $a \to b$ is a pair $(f : x \to x, \theta : f \Rightarrow \mathbb{M}(a, b))$.
- For any map $f : x \to y$ in $\mathcal{B}$, we get a function $P_{\mathbb{M},A}f : A_y \to A_x$ defined by : $P_{\mathbb{M},A}f(a)$ is the element of A which represents the singleton $\mathbb{M}(-, a)f$.

Then this can be made into a functor $P : \mathbf{Cat}_\kappa \to [\mathbf{Map}(\mathcal{B})^{\mathrm{op}}, \mathbf{Cat}]$.

The $\mathcal{B}$-category of elements

- Now we construct a functor $\int : [\mathbf{Map}(\mathcal{B})^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}(\mathcal{B})$.
- As the resulting category is not necessarily complete or skeletal, we will need some "completion" functor.

Proposition

Let $F : \mathbf{Map}(\mathcal{B})^{\mathrm{op}} \to \mathbf{Cat}$ be an indexed category. Consider the following data :

- The $\mathrm{Ob}(\mathcal{B})$-typed set $\int F$ given by $(\int F)_x = F(x)$ for all $x \in \mathrm{Ob}(\mathcal{B})$.
- For all $a \in F(x)$, $b \in F(y)$ with $x, y \in \mathrm{Ob}(\mathcal{B})$, we define $\mathbb{N}(a, b)$ as the colimit of those $y \to x$ such that there is an arrow $F(f)(a) \to b$ in $F(y)$; we also take 2-cells between those arrows.

Then $(\mathbb{N}, \int F)$ is a $\mathcal{B}$-category.

The Cauchy functor

Proposition

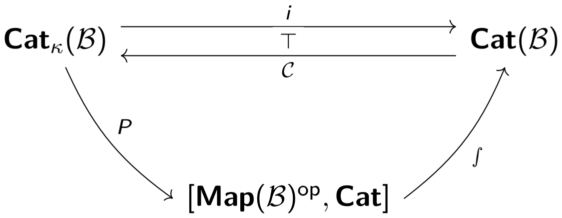
For $(\mathbb{M}, A)$ a $\mathcal{B}$-category, consider the set $\mathcal{C}A$ of singletons of A, i.e. of distributors $\sigma_! : (\mathrm{id}_*, *) \to (\mathbb{M}, A)$ having a right adjoint $\sigma^!$. Type it through $t\sigma = *$, and consider the endomatrix $\mathbb{S}$ on $\mathcal{C}A$ given by : $\mathbb{S}(\sigma_1, \sigma_2) = \sigma_1^!(\sigma_2)_!$.
Then $(\mathbb{S}, \mathcal{C}A)$ is a skeletal complete $\mathcal{B}$-category.

Definition

We define the Cauchy functor $\mathcal{C} : \mathbf{Cat}(\mathcal{B}) \to \mathbf{Cat}_\kappa(\mathcal{B})$ sending $(\mathbb{M}, A)$ to $(\mathbb{S}, \mathcal{C}A)$ and sending any $\mathcal{B}$-functor $F : (\mathbb{M}, A) \to (\mathbb{N}, C)$ to the $\mathcal{B}$-functor $\mathcal{C}F : (\mathbb{S}_A, \mathcal{C}A) \to (\mathbb{S}_B, \mathcal{C}C)$ which sends any singleton σ of $(\mathbb{M}, A)$ to the singleton $F\sigma = (F_!\sigma_!, \sigma^! F^!)$ of $(\mathbb{N}, C)$, where $F_!(c, a) = \mathbb{N}(c, F(a))$ and $F^!(a, c) = \mathbb{N}(F(a), c)$.

The whole picture

- There is an adjunction $\mathcal{C} \dashv i$, where $i : \mathbf{Cat}_\kappa(\mathcal{B}) \to \mathbf{Cat}(\mathcal{B})$ is the inclusion functor.
- More globally, we give here a diagram of all the functors we just defined :



Complete $\mathcal{B}$-categories as a lex reflection

Conjecture

$\mathbf{Cat}_\kappa(\mathcal{B})$ is a left-exact reflective subcategory of $[\mathbf{Map}(\mathcal{B})^{\mathsf{op}}, \mathbf{Cat}]$

- To prove it we must show that we have an adjunction $\mathcal{C}\int \dashv P$. The details would be given by the following :
- The unit of the adjunction would be, for any $\mathcal{F} : \mathbf{Map}(\mathcal{B})^{\mathsf{op}} \to \mathbf{Cat}$, $\eta_\mathcal{F} : \mathcal{F} \to P\mathcal{C}\int\mathcal{F}$. For any $x \in \mathrm{Ob}(\mathcal{B})$, the corresponding functor $\mathcal{F}(x) \to P\mathcal{C}\int\mathcal{F}(x)$ is given by $\eta_\mathcal{F}(a) = \mathbb{N}(-, a)$.
- The counit of the adjunction would be the identity, because the completion of a complete $\mathcal{B}$-category is the original $\mathcal{B}$-category.
- We also need to prove that $\mathcal{C}\int$ preserves finite limits.

Perspectives

- As several attempts have already been made to define enriched sheaves [4], notably as left-exact reflections of $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ in the case of a monoidal-enriched category $\mathcal{C}$. We shall investigate how the definitions we gave relate to that theory ; we shall in particular be interested with the case $\mathcal{V} = \mathbf{Ab}$.
- Notice that we have the same "matricial formalism" that there was for quantaloids, with two operations given by the coproduct (generalizing the sum/join) and composition (generalizing the product of matrices).

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