Internal languages of diagrams of toposes

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The motivating question

Question

What is the internal language of a diagram of toposes, geometric morphisms, and transformations?

Outline

The traditional internal language

The traditional Mitchell–Bénabou internal language of a topos is a higher-order logic:

Syntax	Interpretation in topos
Type A	Object A
Product type $A \times B$	Cartesian product $A \times B$
Function type $A \to B$	Exponential object B^A
Term $f(x, g(y)) : C$ in context $x : A,\ y : D$	Composite morphism $A \times D \xrightarrow{1 \times g} A \times B \xrightarrow{f} C$
Proposition $\varphi(x, y)$ in context $x : A,\ y : B$	Subobject $\varphi \rightarrowtail A \times B$
Conjunction $\varphi \wedge \psi$	Pullback $\varphi \times_{A \times B} \psi$
Implication $\varphi \Rightarrow \psi$	Heyting operation $\varphi \Rightarrow \psi$

Towards dependent type theory

HOL is sufficient to encode nearly all set-based mathematics, but it is sometimes awkward.

Example

In HOL we can only define a category with a single type of arrows:

- two types C_0 and C_1 with $s, t : C_1 \rightrightarrows C_0$, etc.

But in mathematics we often prefer many types of arrows:

- a type C_0 and a family of types $\hom_C(x, y)$ for $x, y : C_0$, etc.

This requires using a dependent type theory a la Martin-Löf instead.

(Dependent type theory also seems to be necessary to formulate usable internal languages for higher toposes.)

The dependently typed internal language

In dependent type theory, instead of just propositions depending on a context, we have arbitrary types depending on a context.

Syntax	Interpretation in topos
Type $C(x, y)$ in context $x : A$, $y : B$	Object C of $\mathcal{E}/(A \times B)$
Type $C(x, y)$ in context $x : A$, $y : B(x)$	Object C of $\mathcal{E}/B$, where $B \in \mathcal{E}/A$
Dependent function type $(x : A) \to B(x)$	$\Pi_A B \in \mathcal{E}$, where $A^* \dashv \Pi_A$.
Equality type $x = y$, for $x, y : A$	Diagonal $\Delta_A : A \to A \times A$, as object of $\mathcal{E}/(A \times A)$

The propositions over A are those types B over A such that $\Pi_{(B \times_A B)} \Delta_B$, syntactically $(x : A)(y : B(x))(z : B(x)) \to (y = z)$.

Universes

In HOL, propositions (in some context) are equivalent to terms (in that context) belonging to the subobject classifier Ω.

In dependent type theory, types in some context can be classified by terms in that context belonging to a universe type $\mathcal{U}$.

- For Russellian paradox reasons, we can't have all types belonging to one universe.
- Requires the topos to have universe objects.
  (All Grothendieck and realizability toposes do.)
- Can perfectly well have dependent types without any universes.

I have no more to say about universes today.

Interpreting dependent type theory in a topos

The interpretation function from syntax to a topos is defined by induction. But for dependent type theory it is more complicated:

- Substitution into types corresponds to pullback:

  If $f : A \to B$, and $C(y)$ is a type in context $y : B$, then $C(f(x))$ in context $x : A$ represents the pullback $f^*(C)$.

- However, substitution in syntax is strictly functorial, while pullback is only pseudofunctorial.

Coherence theorems

We use an intermediate structure:

Definition

A comprehension category is:

- A category $\mathcal{E}$ (whose objects are "contexts")
- A pseudofunctor $\mathcal{T} : \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$ (objects of $\mathcal{T}(\Gamma)$ are "types")
- A universal way to extend $\Gamma \in \mathcal{E}$ by $A \in \mathcal{T}(\Gamma)$ to a new context $\Gamma.A \in \mathcal{E}$.

A topos defines a comprehension category with $\mathcal{T}(\Gamma) = \mathcal{E}/\Gamma$ and functorial action by pullback. Then we use a coherence theorem to make the pseudofunctor strict.

- Seely, "Locally cartesian closed categories and type theory", 1984
- Hofmann, "On the interpretation of type theory in locally cartesian closed categories", 1995
- Lumsdaine–Warren, "The local universes model: an overlooked coherence construction for dependent type theories", 2015

Outline

Diagrams of toposes

Suppose given a diagram of toposes and geometric morphisms: a (pseudo)functor $\mathcal{E} : \mathcal{M} \to \text{Topos}$, where $\mathcal{M}$ is some 2-category.

Example (an $\mathcal{S}$-topos)

A pair of toposes $\mathcal{E}, \mathcal{S}$ and a single geometric morphism $\mathcal{E} \to \mathcal{S}$.

Example (a local $\mathcal{S}$-topos)

A geometric morphism $f : \mathcal{E} \to \mathcal{S}$ with a left adjoint $c : \mathcal{S} \to \mathcal{E}$ whose unit $fc \to 1$ is an isomorphism. (Then $f^* \dashv f_* = c^* \dashv c_*$.)

Example (a totally connected $\mathcal{S}$-topos)

A geometric morphism $f : \mathcal{E} \to \mathcal{S}$ with a right adjoint $d : \mathcal{S} \to \mathcal{E}$ whose counit $1 \to fd$ is an isomorphism. (Then $d^* \dashv d_* = f^* \dashv f_*$.)

Is there an internal language for diagrams?

Modal logic

Traditional modal logic introduces new unary propositional operations called modalities:

$$\Box\varphi = \text{“}\varphi \text{ is necessary”} \qquad \Diamond\varphi = \text{“}\varphi \text{ is possible”}$$

Theorem (Awodey–Birkedal 2001)

Given a local geometric morphism $f : \mathcal{E} \to \mathcal{S}$ with left adjoint $c : \mathcal{S} \to \mathcal{E}$, we can interpret traditional modal logic:

Types	*Objects of $\mathcal{S}$*
Propositions in context $x : A$	*Subobjects of f^*A in $\mathcal{E}$*
$\Box$ *and* $\Diamond$	f^*f_* *and* $c_*f_* = c_*c^*$

- $\Box$ = topological interior, or relative discrete coreflection.
- $\Diamond$ = topological closure, or relative indiscrete reflection.

Towards modal type theories

However, we may want to reason about objects of $\mathcal{E}$ as well, and the actions of f and c on arbitrary objects.

A modal dependent type theory has modalities that act on types rather than propositions.

Say we let types represent objects of $\mathcal{E}$. So we should have:

- A discrete coreflection $\Box$ acting on types, and
- An indiscrete reflection $\Diamond$ acting on types.

Central question

If B is a type in context $x : A$, what is the context of $\Box B$?

Contexts in modal type theory

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The naïve answer is $x' : \Box A$. But this is technically problematic:

1. If $B(x)$ depends on $x : A$, then $\Box(B(x))$ no longer uses x, but some new variable $x' : \Box A$. How do we write it?
2. If $f : C \to \Box A$, substituting $f(y)$ for x' in $\Box B$ yields something that can't be obtained directly as $\Box$ of anything.
3. For implementing a type-theory-based proof assistant, the typechecker wants to ask a different question anyway!

Better question

For $\Box B$ to be a type in context $x : A$, in what context does B have to be a type?

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Better question

For $\Box B$ to be a type in context $x : A$, in what context does B have to be a type?

Spatial type theory

We allow contexts to contain two kinds of variables:

- $x : A$ is an ordinary variable of type A.
- $x :: A$ is a "crisp" variable, semantically the same as $x : \Box A$.

If Γ is such a context, $\Gamma/_\Box$ denotes only its crisp variables.

Now we can answer the typechecker:

For $\Box B$ to be a type in context Γ,
it suffices for B to be a type in context $\Gamma/_\Box$.

Example

If $\Gamma = (x :: A, y : C)$, categorically $\Gamma = \Box A \times C$, so

$$\Gamma/_\Box = (x :: A) = \Box A.$$

Then if $B \in \mathcal{E}/\Box A$, we have $\Box B \in \mathcal{E}/\Box\Box A \simeq \mathcal{E}/\Box A$, which can be pulled back to $\mathcal{E}/(\Box A \times C)$.

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Spatial type theory II

Dually, $\Gamma/_{\lozenge}$ denotes Γ with all variables made crisp.

For $\lozenge B$ to be a type in context Γ,
it suffices for B to be a type in context $\Gamma/_{\lozenge}$.

Example

If $\Gamma = (x :: A, y : C) = \Box A \times C$, then

$$\Gamma/_{\lozenge} = (x :: A, y :: C) = \Box(A \times C).$$

If $B \in \mathcal{E}/\Box(A \times C)$, we have $\lozenge B \in \mathcal{E}/\lozenge\Box(A \times C) \simeq \mathcal{E}/\lozenge(A \times C)$, which can be pulled back to $\mathcal{E}/(\Box A \times C)$.

This gives an internal language for a local geometric morphism.

- Pfenning–Davies, "A judgmental reconstruction of modal logic", 2001
- Licata-S., "Adjoint logic with a 2-category of modes", 2016
- S., "Brouwer's fixed-point theorem in real-cohesive homotopy type theory", 2018

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Spatial type theory II

Dually, $\Gamma/_{\Diamond}$ denotes Γ with all variables made crisp.

For $\Diamond B$ to be a type in context Γ,
it suffices for B to be a type in context $\Gamma/_{\Diamond}$.

Example

If $\Gamma = (x :: A, y : C) = \Box A \times C$, then

$$\Gamma/_{\Diamond} = (x :: A, y :: C) = \Box(A \times C).$$

If $B \in \mathcal{E}/\Box(A \times C)$, we have $\Diamond B \in \mathcal{E}/\Diamond\Box(A \times C) \simeq \mathcal{E}/\Diamond(A \times C)$, which can be pulled back to $\mathcal{E}/(\Box A \times C)$.

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Multimodal type theory

For a local geometric morphism, $\mathcal{S}$ is "included" in $\mathcal{E}$, so it makes sense to focus the type theory on $\mathcal{E}$ with its induced endofunctors.

For a general diagram $\mathcal{M} \to \mathrm{Topos}$, we need a separate mode of types for each object $p \in \mathcal{M}$ (hence each topos $\mathcal{E}_p$).

- Each mode has its type theory, an internal language for $\mathcal{E}_p$.
- Each $\mu : p \to q$ in $\mathcal{M}$ gives a modality $\Box_\mu$ that makes p-types into q-types.

For $\Box_\mu B$ to be a q-type in q-context Γ,
it suffices for B to be a p-type in p-context $\Gamma/_\mu$.

But what is $\Gamma/_\mu$?

For that matter, what is a context?

Inductively generated contexts

In ordinary Martin-Löf dependent type theory, the contexts are inductively generated by:

- There is an empty context.
- If Γ is a context, A is a type in context Γ, and x is a fresh variable, then $\Gamma, (x : A)$ is a context.

In MTT (Gratzer–Kavvos–Nuyts–Birkedal 2021) for diagrams on a 2-category $\mathcal{M}$, the contexts are inductively generated by:

- There is an empty context at each mode p.
- If $\mu : p \to q$ and Γ is a q-context, then $\Gamma/_{\mu}$ is a p-context.
- If $\mu : p \to q$, while Γ is a q-context, A is a p-type in context $\Gamma/_{\mu}$, and x is a fresh variable, then $\Gamma, (x :^{\mu} A)$ is a q-context.

$x :^{\mu} A$ generalizes crisp variables; semantically it means $x : \Box_{\mu} A$.

Variables

(This is the most syntactically technical slide — feel free to zone out.)

In spatial type theory, $/_\square$ and $/_\lozenge$ are operations on contexts.
In MTT, $/_\mu$ is an inductive constructor of contexts.

The "meaning" of $/_\mu$ is defined by the rule for using variables:

If $\Gamma, (x :^\mu A), \Theta$ is a context, and ν is the composite of all the divisions in Θ, we can use the variable x whenever we have a 2-cell $\alpha : \mu \Rightarrow \nu$ in $\mathcal{M}$.

This explains why $/_\mu$ can't be an operation on contexts in general: we have to keep μ around so later (in a further extended context) we can choose 2-cells correctly. If there is a unique choice of μ in all cases, or even a universal one (a "left lifting"), we can make that choice right away; but in general this isn't possible.

Modalities versus divisions

tl;dr

In MTT, each $\mu : p \to q$ in $\mathcal{M}$ induces both:

- An operation $\square_\mu$ from p-types to q-types, and
- An operation $/_\mu$ from q-contexts to p-contexts,

and we can't get rid of the second one.

How are these related? The rule for terms of $\square_\mu A$ is similar:

To have $\mathsf{box}(b) : \square_\mu B$ *in* q*-context* Γ,
it suffices to have $b : B$ *in* p*-context* $\Gamma/_\mu$.

Roughly, this means that $/_\mu$ is left adjoint to $\square_\mu$:

$$\frac{\Gamma/_\mu \vdash_p b : B}{\Gamma \vdash_q \mathsf{box}(b) : \square_\mu B}$$

Interpreting MTT

Theorem (GKNB)

We can interpret MTT over $\mathcal{M}$ in any diagram $\mathcal{E} : \mathcal{M} \to \mathsf{Cat}$ where each category $\mathcal{E}_p$ is a topos and each functor $\mathcal{E}_\mu$ has a left adjoint.

Corollary

If $\mathcal{M} = \mathcal{L}[\mathcal{L}^*]$ is a 2-category $\mathcal{L}$ with a left adjoint freely adjoined for every morphism, we can interpret MTT over $\mathcal{M}$ in any diagram $\mathcal{E} : \mathcal{L} \to \mathsf{Topos}$ where each geometric morphism $\mathcal{E}_\mu$ is essential.

Proof of Corollary.

Since each geometric morphism is an adjunction, $\mathcal{E} : \mathcal{L} \to \mathsf{Topos}$ induces $\mathcal{E} : \mathcal{M} = \mathcal{L}[\mathcal{L}^*] \to \mathsf{Cat}$.
Each direct image μ_* has a left adjoint μ^* to be $/_{\mu_*}$, while essentiality gives a further left adjoint $\mu_!$ to be $/_{\mu^*}$. □

Outline

Avoiding essentiality

What can we do if our geometric morphisms are not essential? The key ideas are:

1. The left adjoint $/_\mu$ only needs to be defined so as to act on the contexts, not the types; whereas
2. The "important part" of the internal language, which tells us something about toposes, involves the types, not the contexts.

Recall that when making a topos into a comprehension category, we used the topos $\mathcal{E}$ itself as the contexts, with $\mathcal{T}(\Gamma) = \mathcal{E}/\Gamma$.

This leaves us some unused freedom: we can alter the category of contexts as long as we "don't change" the categories of types.

Presheaves as contexts for sheaves

Let j be a topology on a topos $\mathcal{E}$, with subcategory of sheaves $\mathcal{E}_j \subseteq \mathcal{E}$ and sheafification functor $L_j : \mathcal{E} \to \mathcal{E}_j$.

Definition

A morphism $p : B \to A$ in $\mathcal{E}$ is a relative sheaf if the following square is a pullback:

$$\begin{array}{ccc} B & \longrightarrow & L_j B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle L_j p} \\ A & \longrightarrow & L_j A \end{array}$$

If $\mathcal{T}_j(A) \subseteq \mathcal{E}/A$ denotes the category of relative sheaves, we have an equivalence of categories $\mathcal{T}_j(A) \simeq \mathcal{E}_j / L_j A$.

Thus, $(\mathcal{E}, \mathcal{T}_j)$ is a comprehension category "equivalent" to $\mathcal{E}_j$.

Presenting geometric morphisms by essential ones

Let $f : \mathcal{E} \to \mathcal{S}$ be a geometric morphism, and suppose we have sites

$$\mathcal{E} = \mathsf{Sh}(\mathbb{D}, k) \qquad \text{and} \qquad \mathcal{S} = \mathsf{Sh}(\mathbb{C}, j)$$

and a cover-reflecting* functor $\ell : \mathbb{D} \to \mathbb{C}$ (same direction!) that induces f as

$$f_* = \mathsf{Ran}_\ell$$
$$f^* = \left(\mathsf{Sh}(\mathbb{C}, j) \hookrightarrow \mathsf{Psh}(\mathbb{C}) \xrightarrow{\ell^*} \mathsf{Psh}(\mathbb{D}) \xrightarrow{L_k} \mathsf{Sh}(\mathbb{D}, k)\right).$$

Then f is "presented" by an essential geometric morphism between presheaf categories:

$$\mathsf{Lan}_\ell \dashv \ell^* \dashv \mathsf{Ran}_\ell$$
$$\mathsf{Psh}(\mathbb{D}) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathsf{Psh}(\mathbb{C})$$

* also known as "covering lifting" or a "comorphism of sites".

Finding cover-reflecting functors, part 0

Given $f : \mathcal{E} \to \mathcal{S}$, our goal now is to find sites for $\mathcal{E}$ and $\mathcal{S}$ such that f is presented by a cover-reflecting functor.

- In Moerdijk, "Continuous fibrations and inverse limits of toposes" (1986) this is achieved by considering $\mathcal{E}$ as an $\mathcal{S}$-topos presented by an internal site in $\mathcal{S}$ and then "externalizing" that internal site relative to some site for $\mathcal{S}$.
- We will use a more explicit comma-category construction, which appears in Caramello, "Denseness conditions, morphisms and equivalences of toposes" (2020), and generalizes better.

Finding cover-reflecting functors, part 1

Given $f : \mathcal{E} \to \mathcal{S}$, in the usual way we can find subcanonical sites with finite limits

$$\mathcal{E} = \mathsf{Sh}(\mathbb{B}, i) \qquad \text{and} \qquad \mathcal{S} = \mathsf{Sh}(\mathbb{C}, j)$$

such that $f^*(\mathbb{C}) \subseteq \mathbb{B}$. Write $f^\dagger = f^*|_{\mathbb{C}} : \mathbb{C} \to \mathbb{B}$ (opposite direction!)

Define $\mathbb{D} = (\mathbb{B} \downarrow f^\dagger)$, with objects $(B \in \mathbb{B}, C \in \mathbb{C}, \phi : B \to f^\dagger C)$.

1. The forgetful functor $u : \mathbb{D} \to \mathbb{B}$ has a fully faithful right adjoint $v(B) = (B, 1, !)$.
2. Thus, $u^* : \mathsf{Psh}(\mathbb{B}) \to \mathsf{Psh}(\mathbb{D})$ is a fully faithful left adjoint.
3. u^* also has a left adjoint Lan_u, which is left exact since u is.
4. So $\mathsf{Psh}(\mathbb{B})$, hence also $\mathsf{Sh}(\mathbb{B}, i)$, is a subtopos of $\mathsf{Psh}(\mathbb{D})$.
5. So there is a topology k on $\mathbb{D}$ such that $\mathcal{E} = \mathsf{Sh}(\mathbb{D}, k)$.

Finding cover-reflecting functors, part 2

Recall $f^\dagger = f^*|_{\mathbb{C}} : \mathbb{C} \to \mathbb{B}$ and $\mathbb{D} = (\mathbb{B} \downarrow f^\dagger)$.

Define $g : \mathbb{C} \to \mathbb{D}$ by $g(C) = (f^\dagger(C), C, 1_{f^\dagger(C)})$. Then:

6. g is left exact and $u \circ g = f^\dagger : \mathbb{C} \to \mathbb{B}$.
7. Since the topology k on $\mathbb{D}$ is created by u, and $f^\dagger$ is cover-preserving, g is also cover-preserving.
8. As u induces an equivalence on sheaf categories, g induces the same geometric morphism as $f^\dagger$ on sheaf categories, namely f.
9. g has a left adjoint $\ell : \mathbb{D} \to \mathbb{C}$ defined by $\ell(B, C, \phi) = C$, which is therefore cover-reflecting and also induces f.

So we can present any geometric morphism by an essential one between presheaf categories, and thus interpret MTT over $\mathbb{2}[\mathbb{2}^*]$ in any single geometric morphism.

Outline

Presenting diagrams

Now, given any diagram $\mathcal{E} : \mathcal{L} \to \text{Topos}$, we want to do the same thing for all the geometric morphisms in its image, simultaneously, in order to interpret MTT over $\mathcal{M} = \mathcal{L}[\mathcal{L}^*]$.

That is, we want to find sites of definition for all the toposes in its image, with respect to which all the geometric morphisms in its image are presented by cover-reflecting functors.

The basic idea is that comma categories generalize to oplax limits.

Step 1: A diagram of sites

Assumption

$\mathcal{E} : \mathcal{L} \to$ Topos is a pseudofunctor, where $\mathcal{L}$ is a finite 2-category.

Write $\mu^* \dashv \mu_*$ for the geometric morphism induced by $\mu \in \mathcal{L}(p, q)$.

Step 1

Find subcanonical sites with finite limits, $\mathcal{E}_p = \mathrm{Sh}(\mathbb{C}_p, j_p)$, such that for each $\mu : p \to q$, we have $\mu^*(\mathbb{C}_q) \subseteq \mathbb{C}_p$.

(E.g. find generators $\mathbb{B}_p \subseteq \mathcal{E}_p$ and let $\mathbb{C}_p$ be the closure of $\bigcup_{\mu : p \to q} \mu^*(\mathbb{B}_q)$ under finite limits.)

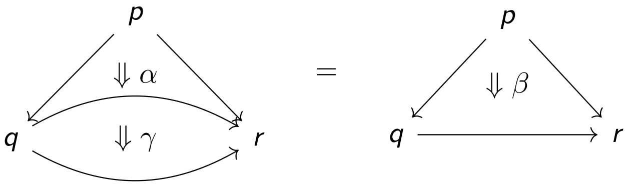
We get a pseudofunctor $\mathbb{C} : \mathcal{L}^{\mathrm{op}} \to$ Lex, where Lex is the 2-category of categories with finite limits and left exact functors.

Write $\mu^\dagger = \mu^*|_{\mathbb{C}_q} : \mathbb{C}_q \to \mathbb{C}_p$ for the functor induced by $\mu \in \mathcal{L}(p, q)$.

Step 2: Oplax limits

Fix $p \in \mathcal{L}$, and let $p /\!\!/ \mathcal{L}$ denote its lax slice 2-category:

- Its objects are pairs (q, μ) where $q \in \mathcal{L}$ and $\mu : p \to q$.
- Its morphisms $(q, \mu) \to (r, \nu)$ are pair (ϱ, α) where $\varrho : q \to r$ and $\alpha : \nu \Rightarrow \varrho \circ \mu$.
- Its 2-cells $(\varrho, \alpha) \Rightarrow (\sigma, \beta)$ are 2-cells $\gamma : \varrho \Rightarrow \sigma$ such that



There is a 2-functor $\pi_p : p /\!\!/ \mathcal{L} \to \mathcal{L}$ with $\pi_p((q, \mu)) = q$.

Step 2: Oplax limits

Let $\mathbb{D}_p$ be the oplax limit of $(p /\!/ \mathcal{L})^{\mathrm{op}} \xrightarrow{\pi_p} \mathcal{L}^{\mathrm{op}} \xrightarrow{\mathbb{C}} \mathrm{Lex}$.

Thus an object of $\mathbb{D}_p$ consists of:

- For each $\mu : p \to q$, an object $\Gamma^\mu \in \mathbb{C}_q$.
- For each $\alpha : \nu \Rightarrow \varrho \circ \mu$, a morphism $\Gamma^\mu \to \varrho^\dagger(\Gamma^\nu)$.
- Functoriality and compatibility axioms for 2-cells.

Example

If $\mathcal{L} = 2 = \{p \xrightarrow{\nu} q\}$, then:

- $p /\!/ \mathcal{L} = \mathcal{L}$ and $\mathbb{D}_p = (\mathbb{C}_p \downarrow \nu^\dagger)$. Its objects have $\Gamma^{1_p} \in \mathbb{C}_p$ and $\Gamma^\nu \in \mathbb{C}_q$, with $\Gamma^{1_p} \to \nu^\dagger(\Gamma^\nu)$ from

- $q /\!/ \mathcal{L} = 1$ and $\mathbb{D}_q = \mathbb{C}_q$.

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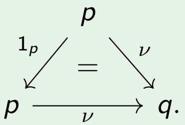
- Functoriality and compatibility axioms for 2-cells.

Example

If $\mathcal{L} = \mathbb{2} = \{p \xrightarrow{\nu} q\}$, then:

- $p /\!\!/ \mathcal{L} = \mathcal{L}$ and $\mathbb{D}_p = (\mathbb{C}_p \downarrow \nu^\dagger)$. Its objects have $\Gamma^{1_p} \in \mathbb{C}_p$ and $\Gamma^\nu \in \mathbb{C}_q$, with $\Gamma^{1_p} \to \nu^\dagger(\Gamma^\nu)$ from



- $q /\!\!/ \mathcal{L} = 1$ and $\mathbb{D}_q = \mathbb{C}_q$.

Step 3: Topologies

Recall an object of $\mathbb{D}_p$ consists of an object $\Gamma^\mu \in \mathbb{C}_q$ for each $\mu : p \to q$, plus morphisms and axioms.

1. There is a forgetful functor $u_p : \mathbb{D}_p \to \mathbb{C}_p$ with $u_p(\Gamma) = \Gamma^{1_p}$.
2. u_p has a right adjoint v_p defined by

$$v_p(A)^{\mu:p\to q} = \lim_{\substack{\sigma:q\to p \\ \beta:1_p\Rightarrow\sigma\circ\mu}} \sigma^\dagger(A)$$

   (Defining the morphisms uses that $\nu^\dagger$ preserves this finite limit.)

3. v_p is fully faithful, since

$$u_p(v_p(A)) = v_p(A)^{1_p} = \lim_{\substack{\sigma:p\to p \\ \beta:1_p\Rightarrow\sigma}} \sigma^\dagger(A) \cong {1_p}^\dagger(A) \cong A$$

4. Therefore, just as before, there is a topology k_p on $\mathbb{D}_p$ such that $\mathrm{Sh}(\mathbb{D}_p, k_p) = \mathrm{Sh}(\mathbb{C}_p, j_p) = \mathcal{E}_p$.

Step 4: Morphisms of sites

5. For $\mu : p \to q$, define $\ell_\mu : \mathbb{D}_p \to \mathbb{D}_q$ by $\ell_\mu(\Gamma)^{\nu:q\to r} = \Gamma^{\nu\circ\mu}$.
6. ℓ_μ has a right adjoint $g_\mu : \mathbb{D}_q \to \mathbb{D}_p$ defined with finite limits, and the following square commutes (up to isomorphism):

$$\begin{array}{ccc} \mathbb{D}_p & \xrightarrow{u_p} & \mathbb{C}_p \\ {\scriptstyle g_\mu}\uparrow & & \uparrow{\scriptstyle \mu^\dagger} \\ \mathbb{D}_q & \xrightarrow[u_q]{} & \mathbb{C}_q \end{array}$$

7. Now as before:
   - Since the topology k_p on $\mathbb{D}_p$ is created by u_p, and $\mu^\dagger : \mathbb{C}_q \to \mathbb{C}_p$ is cover-preserving, g_μ is also cover-preserving.
   - Since u_p and u_q induce equivalences on sheaf categories, g_μ induces the same geometric morphism as $\mu^\dagger : \mathbb{C}_q \to \mathbb{C}_p$, namely $(\mu^* \dashv \mu_*) : \mathcal{E}_p \to \mathcal{E}_q$.
   - Since $\ell_\mu \dashv g_\mu$, the functor ℓ_μ is cover-reflecting and also induces $(\mu^* \dashv \mu_*)$.

Conclusion

Theorem

For any finite 2-category $\mathcal{L}$, we can present any $\mathcal{E} : \mathcal{L} \to \mathsf{Topos}$ by a diagram of sites and cover-reflecting functors, hence by a diagram of presheaf categories and essential geometric morphisms.

Corollary

We can interpret MTT over $\mathcal{M} = \mathcal{L}[\mathcal{L}^*]$ in any such $\mathcal{E}$.

Open questions

- What if $\mathcal{L}$ is infinite? This method only works if each inverse image functor in the diagram preserves $\mathcal{L}$-sized limits.
  - E.g. an idempotent monad or comonad is a finite diagram, but a non-idempotent one is not.
- What about higher toposes? This method works for diagrams of ∞-toposes that are 1-localic, i.e. ∞-sheaves on a 1-site.

Thank you!

Thank you!

https://arxiv.org/abs/2303.02572