TOPOLOGICAL DERIVATORS

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1. Motivation

If one considers small categories as spaces, derivators are forms of (co)homology for spaces. More precisely, derivators are systems of *Grothendieck operations* from where (co)homology theories of spaces arise. In other words, the theory of derivators is the theory of (co)homology theories for small categories.
Extending this previous idea, if one regards now topoi as spaces (in place of small categories), then the *theory* of topological derivators is the theory of (co)homology theories of topoi.

What motivates this extension from small categories to topoi?

- First, Grothendieck achieves the axioms of derivators by examining the construction

$$A \longmapsto \mathrm{Derab}(A)$$

which is actually defined for all topoi

$$X \longmapsto \mathrm{Derab}(X)$$

- Secondly, one can prove that if A and B are two Morita-equivalent small categories, i.e., $\widehat{A} \simeq \widehat{B}$, then $\mathbb{D}(A) \simeq \mathbb{D}(B)$ for every derivator $\mathbb{D}$, which means that derivators are actually determined by the topoi of presheaves associated to small categories, and not by small categories themselves.

Inspired by these two observations, Grothendieck writes the following in a letter to Thomason:

> "...*Cela implique aussi, quand Diag est égal à Cat tout entier, que l'on peut regarder D comme provenant d'un 2-foncteur*
>
> $$Topcat^o \longrightarrow CAT$$
>
> *allant de la 2-catégorie des topos "catégoriques" (i.e. équivalents à un topos provenant d'un X dans Cat) dans la catégorie CAT. Ceci vu, on peut espérer étendre le dérivateur, c'est-à-dire la théorie de coefficients envisagée, à la catégorie Top des topos tout entière, i.e. en un foncteur (qu'on notera encore D)*
>
> $$D : Top^o \longrightarrow CAT$$
>
> *J'ai idée que ça doit être toujours possible, et de façon essentiellement unique. Ça l'est en tous cas dans tous les cas concrets que j'ai regardés.*"

2. Axioms of topological derivators

A topological prederivator is a 2-functor of the form

$$\mathbf{D} : \mathsf{Top}^o \longrightarrow \mathsf{CAT}$$

$$X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y \;\; (\Downarrow \alpha) \qquad \mapsto \qquad \mathbf{D}(Y) \underset{\varphi^*}{\overset{\psi^*}{\rightrightarrows}} \mathbf{D}(X) \;\; (\Downarrow \alpha^*)$$

The functors $\varphi^* =_{df} \mathbf{D}(\varphi)$ associated to geometric morphisms φ are called the *inverse images* of φ (with respect to $\mathbf{D}$).

Definition - A family

$$\rho_i : E_i \longrightarrow X, \qquad i \in I$$

of geometric morphisms between topoi is called a *geometric covering* when the reciprocal family of inverse images

$$(\rho_i)^{-1} : X \longrightarrow E_i, \qquad i \in I$$

is jointly conservative.
Example: If X is a topos with enough mainly points, then there exists a geometric covering of the form

$$\xi : P \longrightarrow X, \qquad \xi \in \Xi.$$

The Theorem of Barr affirms that every topos admits a geometric covering of the form

$$\xi : B \longrightarrow X$$

for some Boolean topos B.

Definition - A topological prederivator $\mathbf{D}$ is called a *left topological derivator* when it verifies the following conditions:

Topder 1. a) For any couple (X, Y) of topoi, the functor

$$(\imath^*, \jmath^*) : \mathbf{D}(X + Y) \longrightarrow \mathbf{D}(X) \times \mathbf{D}(Y)$$

induced from the geometric immersions $\imath : X \to X + Y$ and $\jmath : Y \to X + Y$, is an equivalence of categories.
b) $\mathbf{D}(\varnothing_{top}) \simeq e$.

Topder 2. For any geometric covering of topoi

$$\rho_i : E_i \longrightarrow X, \qquad i \in I,$$

the family of functors

$$(\rho_i)^* : \mathbf{D}(X) \longrightarrow \mathbf{D}(E_i), \qquad i \in I$$

is jointly conservative.

Topder 3g. For any geometric morphism $\varphi : X \to Y$ between topoi, the functor

$$\varphi^* : \mathbf{D}(Y) \longrightarrow \mathbf{D}(X)$$

admits a right adjoint

$$\varphi_* : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y),$$

which is called the *direct cohomological image* of φ. Moreover, if φ is a *geometric immersion*, then φ_* is fully faithful.

Topder 4g. For any geometric morphism $\varphi : X \to Y$ between topoi and any object U of Y, the natural arrow

$$(j_{Y,U})^* \varphi_* \Longrightarrow (\varphi_{/U})_* (j_{X,U})^*$$

associated to the 2-square

$$\begin{array}{ccc} X/U & \xrightarrow{\;j_{X,U}\;} & X \\ {\scriptstyle \varphi_{/U}} \downarrow & & \downarrow {\scriptstyle \varphi} \\ Y/U & \xrightarrow[\;j_{Y,U}\;]{} & Y \end{array}$$

is an isomorphism. Here, $X/U = X/\varphi^{-1}U$ and $j_{Y,U}$ (resp. $j_{X,U}$) denotes the localization geometric morphism.

There is a problem for the existence of direct homological images.

- Given a prederivator $\mathbb{D}$ of domain Cat, one can form the dual prederivator $\mathbb{D}^o$ of $\mathbb{D}$, which is given by the formula:

$$\mathbb{D}^o(A) =_{df} \mathbb{D}(A^o)^o.$$

- Therefore, the theories of left and right derivators of domain Cat are dual to one another: after we define what are left derivators, we can define right derivators as being the prederivators $\mathbb{D}$ such that $\mathbb{D}^o$ is a left derivator. A derivator is a prederivator which is both a right and a left derivator.

- By duality, the main axiom of right derivators is the existence of *direct homological images*, which are *left* adjoints of the inverse images, just as the existence of *direct cohomological images*, i.e., *right* adjoints of inverse images, is the main axiom of left derivators.

- However, topoi are not dualizable in the following sense: given a topos X, the dual category X^o is not a topos in general. Therefore, we can not hope to dualize (as we do for left derivators) in order to obtain the axioms of right topological derivators.

- Yet, direct homological images could eventually exist for *protopoi* (categories of the form $\mathrm{Pro}(X)$ for a topos X).

- We could also consider developing the theory of derivators for a more flexible notion of 'topos', the pseudo-topoi, which are dualizable.

- Anyway, the general existence of direct homological images requires us to go beyond topoi.

In order to maintain ourselves in a topos environment, I propose the following axioms:

Topder 3d. For any topos X and any functor $u : I \to J$ between small categories, the inverse image

$$(u_X)^* : \mathbf{D}(X^J) \longrightarrow \mathbf{D}(X^I)$$

associated to the geometric morphism

$$u_X : X^I \longrightarrow X^J$$

admits a left adjoint

$$(u_X)_! : \mathbf{D}(X^I) \longrightarrow \mathbf{D}(X^J).$$

Topder 4d. Given a geometric morphism $\varphi : X \to Y$ between topoi and a functor $u : I \to J$ between small categories, the natural transformation

$$(u_X)_! \varphi_I^* \Longrightarrow \varphi_J^* (u_Y)_!$$

associated to the 2-square

$$\begin{array}{ccc} X^I & \xrightarrow{\ u_X\ } & X^J \\ {\scriptstyle \varphi_I}\downarrow & & \downarrow{\scriptstyle \varphi_J} \\ Y^I & \xrightarrow[\ u_Y\ ]{} & Y^J \end{array}$$

is an isomorphism.

Definition - A topological prederivator **D** is called a *topological derivator* when it satisfies all the conditions:

Topder1	Topder2
Topder3g	Topder4g
Topder3d	Topder4d

3. Some examples of topological derivators

a) If C is a locally finitely presentable category, then the map $X \mapsto \mathrm{Fais}(X, C)$ defines a topological derivator.

b) For every topos E, there is a topological derivator $? \boxtimes E$:

$$X \boxtimes E \simeq \mathrm{Fais}(X, E) \simeq \mathrm{Fais}(E, X)$$

c) For any topos E and any topological derivator $\mathbf{D}$, the topological derivator defined by the formula:

$$\mathbf{D}_E : X \longmapsto \mathbf{D}_E(X) =_{df} \mathbf{D}(E \boxtimes X)$$

is a topological derivator.

The two main homotopical examples:

d) For any topos X, let $\mathsf{DERAB}(X)$ be the derived (triangulated) category associated to the Grothendieck category $\mathsf{AB}(X)$ of abelian sheaves over X. Then, the map

$$X \longmapsto \mathsf{DERAB}(X)$$

defines a topological derivator.

e) For any topos X, let $\mathsf{HOT}(X)$ be the homotopy category associated to the Illusie-Joyal model structure on simplicial sheaves over X. Then, the map

$$X \longmapsto \mathsf{HOT}(X)$$

defines a topological derivator.

4. Cohomological equivalences

For each topos X, let

$$p_X : X \longrightarrow P =_{df} \mathit{Set}$$

be the essentially unique geometric morphism from X to the *point topos* P (which is just the category of sets).
Let $\mathbf{D}$ be a topological derivator. Given a topos X and an object Ω of $\mathbf{D}(X)$, we define the symbols

$$\mathrm{H}^*_{\mathbf{D}}(X;\Omega) =_{df} (p_X)_*\Omega \qquad \mathrm{H}^{\mathbf{D}}_*(X;\Omega) =_{df} (p_X)_!\Omega$$

to indicate respectively the cohomology and the homology of X with coefficients in Ω with respect to $\mathbf{D}$.

Every geometric morphism $\varphi : X \to Y$ of topoi induces a natural arrow

$$\mathrm{H}^*_{\mathbf{D}}(Y; \Omega) \longrightarrow \mathrm{H}^*_{\mathbf{D}}(X; \varphi^*\Omega)$$

for Ω varying through the objects of $\mathbf{D}(Y)$.

An object Ω of $\mathbf{D}(Y)$ is called *constant* when $\Omega \cong (p_Y)^*\Lambda$ for some object Λ of $\mathbf{D}(P)$.

Definition - A geometric morphism is called a **D**-*equivalence* when the canonical arrow

$$\mathsf{H}^*_{\mathbf{D}}(Y;\Omega) \longrightarrow \mathsf{H}^*_{\mathbf{D}}(X;\varphi^*\Omega)$$

is an isomorphism for every constant object Ω of $\mathbf{D}(Y)$.

More generally, let

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

be an essentially commutative triangle of topoi. Then, one can associate to it a natural arrow of the form

$$\mathrm{H}^*_{\mathbf{D}}(Y; q^*\Lambda) \longrightarrow \mathrm{H}^*_{\mathbf{D}}(X; p^*\Lambda)$$

for Λ varying through the objects of $\mathbf{D}(Z)$. We say that φ is a **D**-*equivalence relative to* Z if the above arrow is always an isomorphism.

Therefore, topological derivators give rise to a class

$$\mathcal{W}_{/X}$$

of arrows at the category

$$\mathrm{Top}/X$$

of X-topoi: the arrows which are *relative* **D**-equivalences for every topological derivator **D**.

On the other hand, there is a notion of *internal weak homotopy equivalence* for every topos X: the Illusie-Joyal equivalences in the category

$$X_{\Delta} =_{df} \mathrm{Fun}(\Delta^{o}, X)$$

of simplicial sheaves over X.

The nature of topological derivators lies in the relation between these two different notions: the *relative cohomological equivalences* and the *internal weak homotopy equivalences*. This relation can be understood via fibred topoi.

A fibred X-topos over a small category I is a pseudo-functor of the form

$$F : I \longrightarrow \mathrm{Top}/X.$$

For every fibred X-topos F over a small category I, there exists a total topos

$$\mathrm{Top}F$$

which comes with two canonical geometric morphisms

$$\theta_F : \mathrm{Top}F \to \widehat{I}, \qquad \pi_F : \mathrm{Top}F \to X,$$

and evaluating geometric morphisms

$$i : F_i \longrightarrow \mathrm{Top}F, \qquad i \in Ob(I).$$

Example a) If X is a fixed topos and we consider the constant fibred topos

$$\underline{X} : I \longrightarrow \mathrm{Top}, \quad i \mapsto X, \quad \alpha \mapsto 1_X,$$

then

$$\mathrm{Top}\underline{X} \simeq X^I.$$

In this case, the evaluating geometric morphisms $i : X \to X^I$ are determined by the inverse images

$$i^{-1} : X^I \longrightarrow X, \quad F \mapsto F_i.$$

b) If I is a discrete small category (a small set) and

$$X : I \longrightarrow \mathrm{Top}$$

is a fibred topos (a small family of topoi indexed by I), then

$$\mathrm{Top}X \simeq \sum_{i \in I} X_i$$

c) If $F : I \to \mathrm{Cat}$ is a functor from I to the category of small categories, then it can be extended to a fibred topos

$$I \longrightarrow \mathrm{Top}, \quad i \mapsto \widehat{F_i}.$$

In this case,

$$\mathrm{Top}F \simeq \widehat{\textstyle\int F},$$

where $\int F$ denotes the Grothendieck integration construction associated to F.

d) If $F : I \to X$ is a diagram in a topos X, then one can see F as a fibred X-topos over I:

$$I \longrightarrow \mathrm{Top}/X, \quad i \mapsto (j_{X,F_i} : X/F_i \to X),$$

and hence, one can associate to F a total X-topos

$$\pi_F : \mathrm{Top}F \longrightarrow X$$

such that the triangle

$$\begin{array}{ccc} X/F_i & \xrightarrow{\quad i \quad} & \mathrm{Top}F \\ & {\scriptstyle j_{X,F_i}} \searrow \quad \swarrow {\scriptstyle \pi_F} & \\ & X & \end{array}$$

commutes for every object $i \in Ob(I)$. Moreover, every arrow $f : F \to G$ in the category X^I induces a commutative triangle

$$\begin{array}{ccc} \mathrm{Top}F & \xrightarrow{\quad \mathrm{Top}(f) \quad} & \mathrm{Top}G \\ & {\scriptstyle \pi_F} \searrow \quad \swarrow {\scriptstyle \pi_G} & \\ & X & \end{array}$$

Theorem - Let X be a topos, $f : F \to G$ be an arrow of $X_\Delta =_{df} \mathrm{Fun}(\Delta^o, X)$ and

$$\begin{array}{ccc} \mathrm{Top}F & \xrightarrow{\mathrm{Top}(f)} & \mathrm{Top}G \\ & {\scriptstyle \pi_F} \searrow \quad \swarrow {\scriptstyle \pi_G} & \\ & X & \end{array}$$

be the geometric morphism of X-topoi associated to f by the totalization of the fibred topoi

$$F : \Delta^o \longrightarrow \mathrm{Top}/X, \quad [n] \mapsto (X/F_n, j_{X,F_n})$$

and

$$G : \Delta^o \longrightarrow \mathrm{Top}/X, \quad [n] \mapsto (X/G_n, j_{X,G_n}).$$

If f is an Illusie-Joyal equivalence in X_Δ, then $\mathrm{Top}(f)$ is a **D**-equivalence relative to X with respect to every left topological derivator **D**.

Thank you very much!