Galois Theory of Differential Schemes

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TOPOSES in MONDOVÌ

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CLASSICAL PICARD-VESSIOT THEORY

$$\begin{tikzcd} (A,\delta_A) \arrow[r, hook] & (L,\delta_L) \\ (K,\delta_K) \arrow[u] \arrow[ur] & \end{tikzcd}$$

PV extension of differential fields with

- PV ring (A,δ_A)
- $k = \mathrm{Const}(K) = \mathrm{Const}(L)$.

$\rightsquigarrow$ PV Galois group $G = \mathrm{Gal}^{PV}(L/K) = \mathrm{Spec}(\mathrm{Const}(A \otimes_K A))$

linear alg. group / k

$$\left\{ \begin{array}{c} \text{intermediate differential} \\ \text{field extensions} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{closed subgroups} \\ \text{of } G \end{array} \right\}$$

JANELIDZE'S CATEGORICAL FORMULATION

CATEGORICAL GALOIS THEORY :

$$\begin{tikzcd} \delta\text{-}\mathrm{Ring}^{op} \arrow[r, phantom, "\simeq"] \arrow[d, bend right, "\mathrm{Const}"'] \arrow[d, phantom, "\dashv"] & \delta\text{-}\mathrm{Aff} \\ \mathrm{Ring}^{op} \arrow[u, bend right, "(-,0)"'] \arrow[r, phantom, "\simeq"] & \mathrm{Aff} \end{tikzcd}$$

$$\begin{tikzcd} X = \mathrm{Spec}(A) \arrow[r, "f"] \arrow[dr] & Y = \mathrm{Spec}(K) \arrow[d] \\ & S = \mathrm{Spec}(k) \end{tikzcd}$$

is GALOIS, NOT L/K.

Categorical Galois group(oid)

$$G = \mathrm{Gal}[f] \simeq \mathrm{Gal}^{PV}(L/K).$$

Self-splitting :

$$X \times_Y X \simeq X \times_{(S,0)} (G,0)$$

Torsor relation

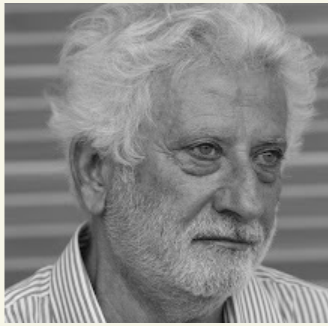
Equivalence :

$$\mathrm{Split}[f] \simeq \mathrm{Aff}^{G}$$

objects in $\delta\text{-}\mathrm{Aff}/_Y$ split by f

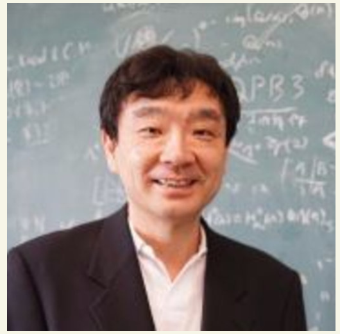
G-actions in Aff.

CARBONI-JANELIDZE-MAGID CORRESPONDENCE



$$\left\{\begin{array}{l}\text{SPLIT}\\ \text{QUOTIENTS of } \begin{array}{c} X\\ \downarrow \\ Y\end{array}\end{array}\right\} \overset{1:1}{\leftrightsquigarrow} \left\{\begin{array}{l}\text{EFFECTIVE}\\ \text{SUBGROUPS of } G\end{array}\right\}$$

$$X/H \longleftarrow\!\shortmid\; H \leq G$$



MASUOKA (Sept 2023): Does Categorical Galois theory recover classical PV correspondence?

Answer: NO! $H \leq G$ is effective in Aff iff G/H affine scheme.

In general G/H can be QUASI-PROJECTIVE.

$\leadsto$ We must work with GENERAL DIFFERENTIAL SCHEMES.

$$\begin{tikzcd} \delta\text{-Sch} \arrow[d, dashed, bend right, "?"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Sch} \arrow[u, bend right, "C = (-, 0)"'] \end{tikzcd}$$

DIFFERENTIAL SCHEMES

S - scheme

S - differential scheme : $(X, (\mathcal{O}_X, \delta_X))$, where $(X, \mathcal{O}_X) \in Sch_{/S}$

$\delta_X \in Der_S(\mathcal{O}_X, \mathcal{O}_X)$.

a vector field on X.

$\rightsquigarrow$ category

$$\delta\text{-}Sch_S$$

Functor

$$C : Sch_{/S} \longrightarrow \delta\text{-}Sch_S$$

$$(X, \mathcal{O}_X) \longmapsto (X, (\mathcal{O}_X, 0)).$$

DIFFERENTIAL SCHEMES AS PRECATEGORY ACTIONS

PRECATEGORY in $\mathrm{Sch}_{/S}$

$$\mathbb{D}(S):\quad S[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_1\varepsilon_2,\varepsilon_2^2) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} S[\varepsilon]/(\varepsilon^2) \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} S$$

Key observation :

$$\delta\text{-}\mathrm{Sch}_S \simeq \left(\mathrm{Sch}_{/S}\right)^{\mathbb{D}(S)}$$

$$(X,\delta_X) \leftrightsquigarrow \mathbb{X} = \left(X_2 \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} X_1 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} X_0\right)$$

CATEGORICAL SCHEME of LEAVES :

$$\pi_0(X,\delta_X) := \pi_0(\mathbb{X}) = \mathrm{coeq}\left(X_1 \rightrightarrows X_0\right)$$

when it exists in Sch.

DIFFERENTIAL DESCENT

$$\mathcal{P} : (Sch/_S)^{op} \longrightarrow CAT \quad \rightsquigarrow \quad \delta\text{-}\mathcal{P} : (\delta\text{-}Sch_S)^{op} \longrightarrow CAT$$

$$(X, \delta_X) \longmapsto \mathcal{P}^{\mathbb{X}}$$

precategory $\mathbb{X}$ actions in $\mathcal{P}$.

$$\begin{tikzcd}
(X,\delta_X) \arrow[d, "f"'] & & \mathbb{X} \arrow[d] & \cdots & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_2"'] & X_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, "f_1"] & X_0 \arrow[d, "f_0"] \\
(Y,\delta_Y) & \rightsquigarrow & \mathbb{Y} & \cdots & Y_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & Y_0
\end{tikzcd}$$

Th f is effective descent for $\delta\text{-}\mathcal{P}$ if f_0 effective descent for $\mathcal{P}$ & f_1 descent for $\mathcal{P}$.

SIMPLICITY & PRECATEGORICAL DESCENT

<u>Def</u> (X, δ_X) is SIMPLE wrt $\mathcal{P} : (Sch_{/S})^{op} \to CAT$

if $\pi_0(X, \delta_X) = \pi_0(\mathbb{X})$ exists and is universal for $\mathcal{P}$.

$$\rightsquigarrow \quad \eta_X : \begin{array}{ccccccc} \mathbb{X} & \cdots & X_2 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & X_1 & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & X_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta(\pi_0(X)) & \cdots & \pi_0(X) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \pi_0(X) & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & \pi_0(X) \end{array}$$

is of PRECATEGORICAL DESCENT,

i.e.

$$\begin{array}{ccc} \mathcal{P}^{\Delta(\pi_0(X))} & & \mathcal{P}^{\mathbb{X}} \\ \parallel & & \parallel \\ C_X : \mathcal{P}(\pi_0(X)) & \longrightarrow & \delta\text{-}\mathcal{P}(X, \delta_X) \\ Q & \longmapsto & \eta_X^* Q \end{array}$$

is fully faithful.

INDEXED FRAMEWORK FOR DIFFERENTIAL GALOIS TH.

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, "\pi_0"'] & \delta\text{-}\mathrm{Sch}_S \\ \mathcal{X} = \mathrm{Sch}/S & \end{tikzcd}$$

that admit π_0

- $\mathcal{P} : \mathcal{X}^{op} \longrightarrow \mathrm{CAT}$

  $\rightsquigarrow \delta\text{-}\mathcal{P} : \mathcal{A}^{op} \longrightarrow \mathrm{CAT}$

- $C : \mathcal{P} \circ \pi_0 \Rightarrow \delta\text{-}\mathcal{P}$ pseudonatural

  $C_Z : \mathcal{P}(\pi_0(Z)) \longrightarrow \delta\text{-}\mathcal{P}(Z)$ as before.

<u>Def</u>

$f : (X, \delta_X) \longrightarrow (Y, \delta_Y) \in \mathcal{A}$ is

pre-PV for $\mathcal{P}$, if :

(1) f is effective descent for $\delta\text{-}\mathcal{P}$

(2) X, $X \times_Y X$, $X \times_Y X \times_Y X$ are simple for $\mathcal{P}$,

<u>Def</u>

class of scheme morphisms

If $U : \mathcal{P} \Rightarrow \mathcal{S}$ suitable,

f is PV for U, if:

(1) — " —

(2) X simple & $f \in \mathrm{Split}[f]$.

GALOIS THEORY OF DIFFERENTIAL SCHEMES

<u>NB</u> • f PV $\Rightarrow$ f pre-PV

• f pre-PV $\Rightarrow$ $\mathbb{G}_f = \left(X \times_Y X \times_Y X \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} X \times_Y X \begin{smallmatrix}\to\\\leftarrow\\\to\end{smallmatrix} X \right) \in \mathrm{Cat}(\mathcal{A})$

$\Rightarrow$ $\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f) \in \mathrm{PreCat}(\mathcal{X})$ GALOIS PRECATEGORY.

• f PV $\Rightarrow$ $\mathrm{Gal}[f] \in \mathrm{Cat}(\mathcal{X})$ GROUPOID.

<u>Th</u> • f pre-PV $\Rightarrow$ equivalence

$$\mathrm{Split}_C[f] \simeq \mathcal{P}^{\mathrm{Gal}[f]}.$$

• f PV $\Rightarrow$ RHS: groupoid actions.

<u>Pf</u> f pre-PV $\Rightarrow$ X, $X \times_Y X$, $X \times_Y X \times_Y X$ simple $\Rightarrow$ C_X, $C_{X \times_Y X}$, $C_{X \times_Y X \times_Y X}$ f.f.

$\Rightarrow$ Janelidze's indexed Galois Th. applies.

APPLICATIONS: QUASI-PROJECTIVE THEORY

- (K, δ_K) diff. field char 0, $k = \mathrm{Const}(K)$, $S = \mathrm{Spec}(k)$,
- $(X, \delta_X) \xrightarrow{f} (Y, \delta_Y) = \mathrm{Spec}(K, \delta_K)$ quasi-projective integral, only leaf is generic pt.
- f is self-split.

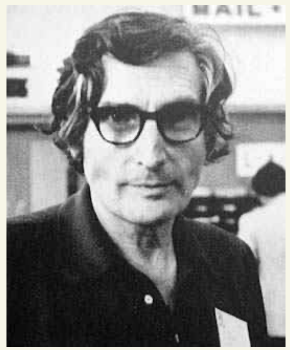
$\Rightarrow$ f is PV, $G = \mathrm{Gal}[f]$ is an S-group scheme, correspondence:

Split quotients

$$\begin{tikzcd} & X \arrow[dl, "\text{fpqc}"'] \arrow[dd, "f"] \\ P \arrow[dr, "\rho"', "\text{quasi-proj.}"] & \\ & Y \end{tikzcd} \quad \longleftrightarrow \quad \begin{matrix} \text{Closed subgroups} \\ G' \leq G \end{matrix}$$

$$X/G' \longleftarrow\!\shortmid \; G'$$

$$P \longmapsto \mathrm{Gal}[X \to P]$$



$\rightsquigarrow$ unifies linear PV theory and STRONGLY NORMAL th. of KOLCHIN.

APPLICATIONS : POLARISED QUASI-PROJECTIVE THEORY

$\mathcal{P} : (\mathrm{Sch}_{/S})^{op} \longrightarrow \mathrm{CAT}$, $\quad \mathcal{P}(V) =$ polarised quasi-projective

$$\begin{array}{c} (Q, \mathcal{L}_Q) \\ {\scriptstyle q}\downarrow \\ V \end{array}$$

$\mathcal{L}_Q$ ← invertible q-ample $\mathcal{O}_Q$-sheaf.

$(X, \delta_X) \xrightarrow{f} (Y, \delta_Y)$ s.t.

(1) (X, δ_X) is simple, $\pi_0(X) = G_0$

(2) $f : X \longrightarrow Y$ is fpqc

(3) $X \times_Y X \simeq X \times_{(G_0, \partial)} (G_1, 0)$ for some $\begin{array}{c} G_1 \\ \downarrow \\ G_0 \end{array}$ quasi-proj.

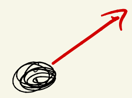
$\Longrightarrow$ f is PV, $\quad \mathrm{Gal}[f] = (G_1 \rightrightarrows G_0)$ groupoid, and have equivalence

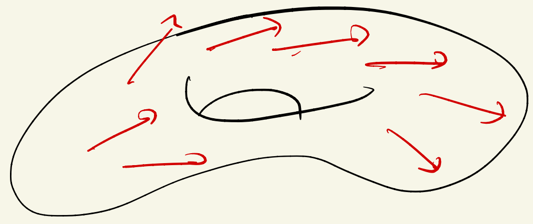
$$\left\{\begin{array}{c} (P, \mathcal{L}_P) \\ \downarrow \\ Y \end{array} \text{polarised quasi-projective } \delta\text{-morphisms split by } f \right\} \rightsquigarrow \left\{ \mathrm{Gal}[f]\text{-actions } \begin{array}{c} (Q, \mathcal{L}_Q) \\ \downarrow \\ G_0 \end{array} \text{polarised} \right\}$$

Differential Galois Theory

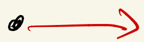
1883 – 2023   2024 –

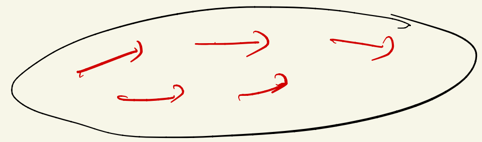
(X,δ_X) :





(Y,δ_Y) :





OUTLOOK :

Differential Galois Theory = (precategorical) Descent + Categorical Galois Theory

$\rightsquigarrow$ difference PV-style Galois theory, more topos-theoretic.

$\rightsquigarrow$ common generalizations.