Presheaves, Sheaves and Sheafification via triposes

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Introduction

Localic toposes

$\mathrm{Sh}(A)$

A is a **locale**

Idea: a locale is a generalization a topological space.

Properties :

- it is a Grothendieck topos
- it is not an instance of ex/lex-completion

Realizability toposes

$\mathrm{RT}(\mathbb{A})$

$\mathbb{A}$ is a **partial combinatory algebra**.

Idea: a pca is a generalization of Kleene's first model.

Properties :

- it is an elementary topos
- it is an instance of ex/lex-completion

Introduction: tripos and tripos-to-topos

Tripos theory

By J. M. E. HYLAND, P. T. JOHNSTONE and A. M. PITTS
Department of Pure Mathematics, University of Cambridge, England

(*Received 28 November* 1979)

Introduction. One of the most important constructions in topos theory is that of the category Shv (A) of sheaves on a locale (= complete Heyting algebra) A. Normally, the objects of this category are described as 'presheaves on A satisfying a gluing condition'; but, as Higgs (7) and Fourman and Scott (5) have observed, they may also be regarded as 'sets structured with an A-valued equality predicate' (briefly, 'A-valued sets'). From the latter point of view, it is an inessential feature of the situation that every sheaf has a canonical representation as a 'complete' A-valued set. In this paper, our aim is to investigate those properties which A *must* have for us to be able to construct a topos of A-valued sets: we shall see that there is one important respect, concerning the relationship between the finitary (propositional) structure and the infinitary (quantifier) structure, in which the usual definition of a locale may be relaxed, and we shall give a number of examples (some of which will be explored more fully in a later paper (8)) to show that this relaxation is potentially useful.

J.M. Hyland, P.T. Johnstone and A.M. Pitts (1980), Tripos theory, *Math. Proc. Camb. Phil. Soc.*

Introduction: tripos and tripos-to-topos

Abstract

The notion of 'tripos' was motivated by the desire to explain in what sense Higgs' description of sheaf toposes as H-valued sets and Hyland's realizability toposes are instances of the same construction. The construction itself can be seen as the universal solution to the problem of realizing the predicates of a first order hyperdoctrine as subobjects in a logos that has all quotients of equivalence relations. In this note it is shown that the resulting logos is actually a topos if and only if the original hyperdoctrine satisfies a certain comprehension property. Triposes satisfy this property, but there are examples of non-triposes satisfying this form of comprehension.

A.M. Pitts (2002), Tripos theory in retrospect, *Math. Struct. in Comp. Science.*

Introduction: tripos and tripos-to-topos

- **Idea:** a **tripos** is a particular Lawvere hyperdoctrine $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$.
- **Tripos**: **T**opos **R**epresenting **I**ndexed **P**artially **O**rdered **S**et
- The **tripos-to-topos** is a construction

$$P \xmapsto{\text{Tr-to-Tp}} \mathrm{T}_P$$

that given a **tripos** $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ produces a topos T_P.

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.

A.M. Pitts (2002), *Tripos theory in retrospect*, Math. Struct. in Comp. Science

Introduction: tripos and tripos-to-topos

Localic tripos

$$A^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$$

A is a **locale**

$$A^{(-)} \xmapsto{\text{Tr-to-Tp}} \mathsf{Sh}(A)$$

Realizability tripos

$$\mathcal{P}_{\mathbb{A}} : \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$$

$\mathbb{A}$ is a **partial combinatory algebra**.

$$\mathcal{P}_{\mathbb{A}} \xmapsto{\text{Tr-to-Tp}} \mathsf{RT}(\mathbb{A})$$

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.
A.M. Pitts (2002), *Tripos theory in retrospect*, Math. Struct. in Comp. Science

Our main contribution

$\mathsf{A}^{(-)} \colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ is a **localic tripos**

$P \colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ is a **tripos**

$$\mathsf{A}^{(-)} \xmapsto{\text{Tr-to-Tp}} \mathsf{Sh}(\mathsf{A})$$

$$P \xmapsto{\text{Tr-to-Tp}} \mathsf{T}_P$$

Our main contribution

$\mathsf{A}^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$ is a **localic tripos**

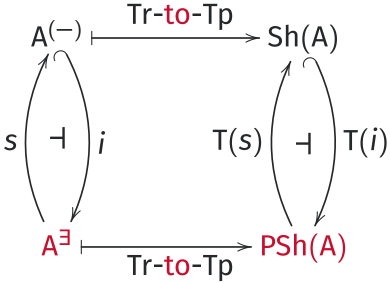
$P : \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$ is a **tripos**

$$\mathsf{A}^{(-)} \xmapsto{\text{Tr-to-Tp}} \mathsf{Sh}(A) \qquad\qquad P \xmapsto{\text{Tr-to-Tp}} \mathsf{T}_P$$

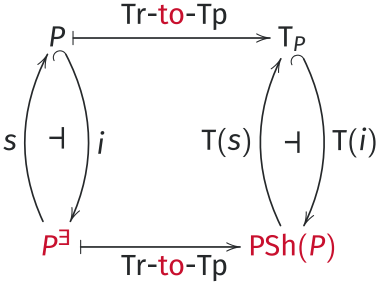
$$\mathsf{Sh}(A) \underset{\dashv}{\rightleftarrows} \mathsf{PSh}(\mathsf{A})$$

Our main contribution

$A^{(-)} : \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is a **localic tripos**



$P : \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is a **tripos**



Generalization to arbitrary-based triposes

The previous approach works in the more general context of arbitrary-based triposes $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$. The two differences are:

- the category $\mathrm{PSh}(P)$ is just an *exact* category;
- $P^{\exists}$ is just an *elementary, existential doctrine*.

We provide a characterization of those triposes such that $\mathrm{PSh}(P)$ is an elementary topos and $P^{\exists}$ is a tripos, and we call them $j^{\exists}$*-triposes*.

Tripos

Definition

A **tripos** is a functor $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ such that

- for every function $f\colon X \longrightarrow Y$ the re-indexing functor $P_f\colon P(Y) \longrightarrow P(X)$ has a left adjoint $\exists_f\colon P(X) \longrightarrow P(Y)$ and a right adjoint $\forall_f\colon P(X) \longrightarrow P(Y)$ in the category Pos, satisfying the Beck-Chevalley condition (BCC);
- there exists a *generic predicate*, namely there exists a set Σ and an element σ of $P(\Sigma)$ such that for every element α of $P(X)$ there exists a function $f\colon X \longrightarrow \Sigma$ such that $\alpha = P_f(\sigma)$.

Examples

Example

Let A be a locale. The representable functor $\mathsf{A}^{(-)} \colon \mathbf{Set}^{op} \longrightarrow \mathbf{Hey}$ assigning to a set X the poset A^X of functions from X to A with the pointwise order is a tripos.

Example

Given a pca $\mathbb{A}$, we can consider the realizability tripos $\mathcal{P}_{\mathbb{A}} \colon \mathbf{Set}^{op} \longrightarrow \mathbf{Hey}$ over Set. For each set X, the partial ordered set $(\mathcal{P}_{\mathbb{A}}(X), \leq)$ is defined as the set of functions $P(\mathbb{A})^X$ from X to the powerset $P(\mathbb{A})$ of $\mathbb{A}$. Given two elements α and β of $\mathcal{P}_{\mathbb{A}}(X)$, we say that $\alpha \leq \beta$ if there exists an element $\overline{a} \in \mathbb{A}$ such that for all $x \in X$ and all $a \in \alpha(x)$, $\overline{a} \cdot a$ is defined and it is an element of $\beta(x)$.

Tripos-to-topos

Tripos-to-topos. Given a tripos $P\colon \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$, the topos T_P consists of:

- **objects:** are pairs (A, ρ) where A is an object of Set and ρ is an element of $P(A \times A)$ satisfying:
  1. *symmetry:* $a_1, a_2 : A \mid \rho(a_1, a_2) \vdash \rho(a_2, a_1)$;
  2. *transitivity:* $a_1, a_2, a_3 : A \mid \rho(a_1, a_2) \wedge \rho(a_2, a_3) \vdash \rho(a_1, a_3)$;
- **arrows:** $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$ are objects ϕ of $P(A \times B)$ such that:
  1. $a : A, b : B \mid \phi(a, b) \vdash \rho(a, a) \wedge \sigma(b, b)$;
  2. $a_1, a_2 : A, b : B \mid \rho(a_1, a_2) \wedge \phi(a_1, b) \vdash \phi(a_2, b)$;
  3. $a : A, b_1, b_2 : B \mid \sigma(b_1, b_2) \wedge \phi(a, b_1) \vdash \phi(a, b_2)$;
  4. $a : A, b_1, b_2 : B \mid \phi(a, b_1) \wedge \phi(a, b_2) \vdash \sigma(b_1, b_2)$;
  5. $a : A \mid \rho(a, a) \vdash \exists b.\phi(a, b)$.

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.
A.M. Pitts (2002), *Tripos theory in retrospect*, Math. Struct. in Comp. Science
M.E. Maietti and G. Rosolini (2013), *Unifying exact completions*, Appl. Categ. Structures.
J. Frey (2015), *Triposes, q-toposes and toposes*, Ann. of Pure and Appl. Logic

Examples

Example

Let A be a locale and the localic tripos $A^{(-)}: \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$. We have the equivalence $T_{A^{(-)}} \equiv \mathrm{Sh}(A)$.

Example

Let $\mathbb{A}$ be a pca, and let us consider the realizability tripos $\mathcal{P}: \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$. We have the equivalence $T_{\mathcal{P}} \equiv \mathrm{RT}(\mathbb{A})$.

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.
A.M. Pitts (2002), *Tripos theory in retrospect*, Math. Struct. in Comp. Science

Triposes and Presheaves

Definition

Let $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ be a tripos. The **Grothendieck category** $\mathcal{G}_P$ of P is given by the following objects and arrows:

- objects are pairs (A, α), where A is an object of Set and $\alpha \in P(A)$;
- a morphism $f\colon (A, \alpha) \longrightarrow (B, \beta)$ is an arrow $f\colon A \longrightarrow B$ of Set such that $\alpha \leq P_f(\beta)$.

We define the category of P-**presheaves** as the category $\mathrm{PSh}(P) := (\mathcal{G}_P)_{\mathrm{ex/lex}}$.

Theorem

Let $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ *be a tripos. The category* $\mathrm{PSh}(P)$ *is an elementary topos.*

Examples

Example

Let A be a locale and the localic tripos $A^{(-)} \colon \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$. We have the equivalence $\mathrm{PSh}(A) \equiv (A_+)_{ex/lex} \equiv (\mathcal{G}_{A^{(-)}})_{ex/lex}$.

Example

Let $\mathbb{A}$ be a pca, and let us consider the realizability tripos $\mathcal{P} \colon \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$. The category $\mathcal{G}_{\mathcal{P}}$ can be described as follows: they are pairs (X, α), where X is a set and $\alpha \subseteq X \times \mathbb{A}$ is a relation. A morphism $f \colon (X, \alpha) \longrightarrow (B, \beta)$ is given by a function $f \colon X \longrightarrow Y$ such that there exists an element $a \in \mathbb{A}$ that tracks f.

$$\mathrm{RT}(\mathbb{A}) \hookrightarrow (\mathcal{G}_{\mathcal{P}})_{ex/lex} \equiv \mathrm{PSh}(\mathcal{P}).$$

Existential completions of triposes

Existential completion. Let $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ be a functor where InfSl in the category of inf-semilattices, i.e. a *primary doctrine*. We can construct a new *existential doctrine*, denoted by $P^{\exists}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$, that is called the **existential completion** of P.

Theorem

Let $P\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ *be a tripos. Then:*

- $P^{\exists}\colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ *is a tripos;*
- $\mathsf{T}_{P^{\exists}} \cong \mathrm{PSh}(P)$.

D. Trotta (2020), *The existential completion*, Theory and Applications of Categories

M.E. Maietti and D. Trotta (2023), *A characterization of generalized existential completions*, Ann. Pure Appl. Log.

Theorem

Let $P\colon \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$ *be a tripos. Then there exists an adjunction of toposes*

$$\mathrm{PSh}(P) \underset{\mathsf{T}(i)}{\overset{\mathsf{T}(s)}{\underset{\bot}{\rightleftarrows}}} \mathsf{T}_P$$

such that $\mathsf{T}(s)\mathsf{T}(i) \cong \mathrm{id}_{\mathsf{T}_P}$.

Corollary

Let $P\colon \mathrm{Set}^{op} \longrightarrow \mathrm{Hey}$ *be a tripos. Then there exists a Lawvere-Tierney topology* $j^{\exists}$ *on* $\mathrm{PSh}(P)$ *such that* $\mathsf{T}_P \equiv \mathrm{Sh}_{j^{\exists}}(\mathrm{PSh}(P))$.

M.E. Maietti and D. Trotta (2023), *A characterization of generalized existential completions*, Ann. Pure Appl. Log.
J. Frey (2015), *Triposes, q-toposes and toposes*, Ann. of Pure and Appl. Logic

Example

Let $A^{(-)} : \mathbf{Set}^{op} \longrightarrow \mathbf{Hey}$ be a localic tripos. The adjunction

$$\mathrm{PSh}(A) \underset{\hookleftarrow}{\overset{\longrightarrow}{\bot}} \mathrm{Sh}(A)$$

is exactly the so-called sheafification.

Example

Let $\mathcal{P}_{\mathbb{A}} : \mathbf{Set}^{op} \longrightarrow \mathbf{Hey}$ be a realizability tripos. Then, we have

$$\mathrm{PSh}(\mathcal{P}_{\mathbb{A}}) \underset{\hookleftarrow}{\overset{\longrightarrow}{\bot}} \mathrm{RT}(\mathbb{A})$$

and hence that $\mathrm{RT}(\mathbb{A}) \equiv \mathrm{Sh}_{j^{\exists}}(\mathrm{PSh}(\mathcal{P}_{\mathbb{A}}))$.

Main references

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