A groupoidal characterisation of theories via topos theory

Joshua Wrigley • Queen Mary University of London

Toposes in Mondovì

On topological groupoids that represent theories, arXiv:2306.16331
Topoi with enough points and topological groupoids, arXiv:2408.15848

Background

Theorem (Ahlbrandt-Ziegler [AZ86])

Let $\mathbb{T}_1, \mathbb{T}_2$ be countably categorical theories.

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Let $M \vDash \mathbb{T}_1$ and $N \vDash \mathbb{T}_2$ be the unique countable models.

There is a homeomorphism of topological groups

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(N),$$

if and only if $\mathbb{T}_1$ and $\mathbb{T}_2$ are *bi-interpretable*.

- A structure is interpretable in another if it can be obtained as a definable quotient of definable subsets.

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This has facilitated applications to model theory from

- combinatorial group theory,
- group cohomology,
- ...

(see the survey article of MacPherson [Ma11]).

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But the groupoid $\mathbf{G}(\mathbb{T})$ is not groupoid of models for $\mathbb{T}$.

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Theorem template

Let $\mathbb{T}_1, \mathbb{T}_2$ be theories with representing groupoids $\mathbb{X}, \mathbb{Y}$.

Then $\mathbb{T}_1, \mathbb{T}_2$ are *Morita equivalent* if and only if $\mathbb{X}, \mathbb{Y}$ are *Morita equivalent*.

- Theories $\mathbb{T}_1, \mathbb{T}_2$ are Morita equivalent if $\mathcal{E}_{\mathbb{T}_1} \simeq \mathcal{E}_{\mathbb{T}_2}$,
- Groupoids $\mathbb{X}, \mathbb{Y}$ are Morita equivalent if $\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\mathbb{Y})$.

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This can be visualised as a 'bridge':

When does $\mathcal{E}_{\mathbb{T}_1} \simeq \mathbf{Sh}(\mathbb{X})$, $\mathcal{E}_{\mathbb{T}_2} \simeq \mathbf{Sh}(\mathbb{Y})$?

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① Under classical assumptions,

$\mathbb{T}_1, \mathbb{T}_2$ are bi-interpretable $\iff$ $\mathbb{T}_1, \mathbb{T}_2$ are Morita equivalent,

see McEldowney [Mc20].

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③ This is the subject of today's presentation.

Main result and overview

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Two *logical groupoids* $\mathbb{X}, \mathbb{Y}$ have equivalent *sheaf topoi* if and only if there exist embeddings

$$\mathbb{X} \subseteq \mathbb{W} \supseteq \mathbb{Y}$$

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that are *weak equivalences*.

I. We recall the construction of the *topos of sheaves* on a topological groupoid.
II. We define the class of *logical groupoids*.
III. We identify the class of *weak equivalences*.

Topological groupoids

Definition

A topological groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ consists of a groupoid

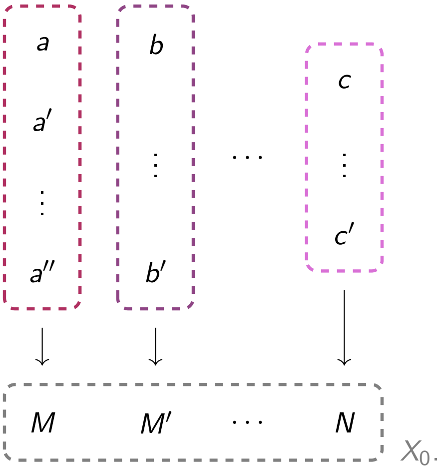
$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \underset{s}{\overset{t}{\underset{\longleftarrow e}{\rightrightarrows}}} X_0, \qquad i : X_1 \to X_1$$

where X_0 and X_1 are endowed with topologies making all the above maps continuous.

If s (equivalently, t) is open, we say $\mathbb{X}$ is an *open* topological groupoid.

Equivariant sheaves on a groupoid

Given a groupoid $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$,



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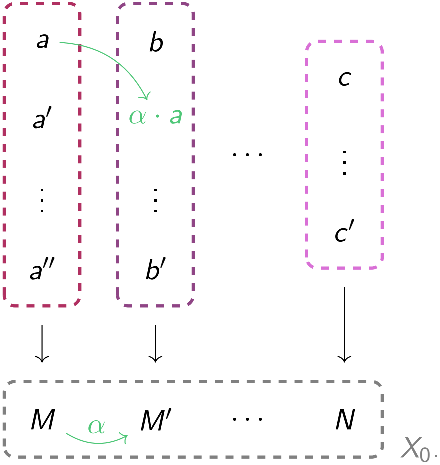

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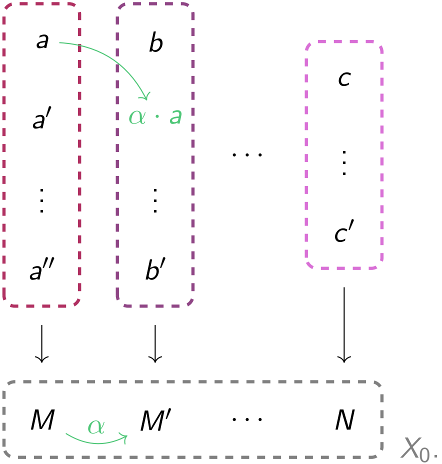

If $\mathbb{X}$ is endowed with topologies, we say that a bundle is a *sheaf* if

(i) $q\colon Y \to X_0$ is a local homeomorphism,

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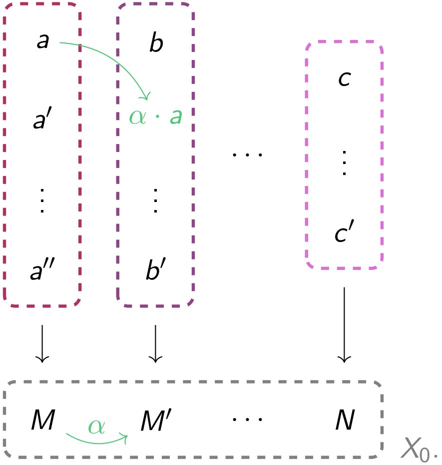

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$$f\colon (Y, q, \beta) \to (Y', q', \beta')$$

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$$q' \circ f = q \text{ and } \alpha \cdot f(y) = f(\alpha \cdot y).$$

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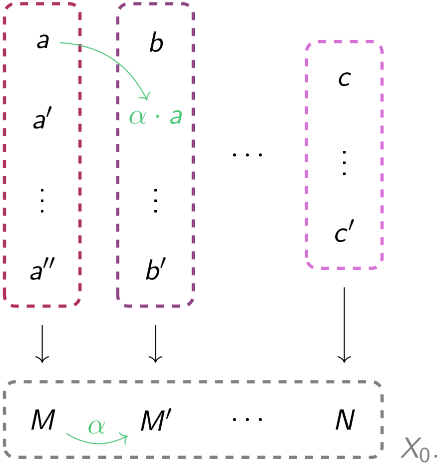

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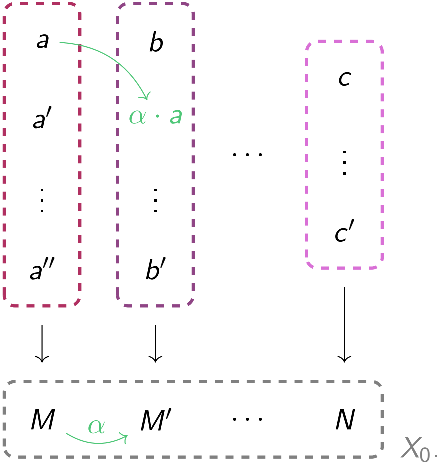

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Example

A topological group G is a topological groupoid.

Its sheaves is the topos $\mathbf{B}G$ of continuous actions by G on discrete sets.

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(iii) There exists a set X such that $G \subseteq \Omega(X)$.

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Definition

We say that a topological groupoid

$$\mathbb{X} = (X_1 \rightrightarrows X_0)$$

is *logical* if

- $\mathbb{X}$ is an open topological groupoid,
- both X_0 and X_1 are T_0 spaces,
- and the topology on X_1 is the coarsest topology determined by $\mathbf{Sh}(\mathbb{X})$.

Weak equivalences

Let $\mathbb{X}$ be a topological groupoid, and let $\mathbb{Y}, \mathbb{U} \subseteq \mathbb{X}$ be subgroupoids.

Each arrow $\alpha \in X_1$ comes with a left $\mathbb{Y}$-action and a right $\mathbb{U}$-action:

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The *bi-orbit* for these actions is the set

$$_{\mathbb{Y}}[\alpha]_{\mathbb{U}} = \{ \beta \circ \alpha \circ \gamma \mid \beta \in Y_1, \gamma \in U_1 \}.$$

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The inclusion $\mathbb{Y} \subseteq \mathbb{X}$ yields an equivalence

$$\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{X})$$

if and only if, for each open subgroupoid $\mathbb{U} \subseteq \mathbb{X}$, the map

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- ${}_{\mathbb{Y}}[s^{-1}(U) \cap t^{-1}(Y_0)]_{\mathbb{U}} \subseteq {}_{\mathbb{Y}}[X_1]_{\mathbb{U}}$ is the subspace of bi-orbits

  $${}_{\mathbb{Y}}[x \xrightarrow{\alpha} y]_{\mathbb{U}}$$

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- ▷ ${}_{\mathbb{X}}[s^{-1}(U)]_{\mathbb{U}} \subseteq {}_{\mathbb{X}}[X_1]_{\mathbb{U}}$ is the subspace of bi-orbits

$${}_{\mathbb{X}}[x \xrightarrow{\alpha} y]_{\mathbb{U}}$$

where $x \in U_0$.

Weak equivalences

Let $\mathbb{X}$ be a topological groupoid, and let $\mathbb{Y}, \mathbb{U} \subseteq \mathbb{X}$ be subgroupoids.

Each arrow $\alpha \in X_1$ comes with a left $\mathbb{Y}$-action and a right $\mathbb{U}$-action:

$$(\beta, \alpha) \mapsto \beta \circ \alpha, \; (\alpha, \gamma) \mapsto \alpha \circ \gamma,$$

where $\beta \in Y_1$ and $\gamma \in U_1$.

The *bi-orbit* for these actions is the set

$${}_{\mathbb{Y}}[\alpha]_{\mathbb{U}} = \{ \beta \circ \alpha \circ \gamma \mid \beta \in Y_1, \gamma \in U_1 \}.$$

The *space of bi-orbits* can be endowed with the quotient topology via the map:

$$X_1 \twoheadrightarrow {}_{\mathbb{Y}}[X_1]_{\mathbb{U}}.$$

Theorem (W.)

Let $\mathbb{X}$ be a logical groupoid and let $\mathbb{Y} \subseteq \mathbb{X}$ be a subgroupoid.

The inclusion $\mathbb{Y} \subseteq \mathbb{X}$ yields an equivalence

$$\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{X})$$

if and only if, for each open subgroupoid $\mathbb{U} \subseteq \mathbb{X}$, the map

$$\begin{aligned} {}_{\mathbb{Y}}[s^{-1}(U) \cap t^{-1}(Y_0)]_{\mathbb{U}} &\to {}_{\mathbb{X}}[s^{-1}(U_0)]_{\mathbb{U}} \\ {}_{\mathbb{Y}}[\alpha]_{\mathbb{U}} &\mapsto {}_{\mathbb{X}}[\alpha]_{\mathbb{U}} \end{aligned}$$

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- ▷ A continuous map is a *quasi-homeomorphism* if the inverse image map is a bijection on open subsets.

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A subgroupoid inclusion $\mathbb{Y} \subseteq \mathbb{X}$ satisfying the theorem is said to be a *weak equivalence*.

Weak equivalences

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Definition

A subgroupoid inclusion $\mathbb{Y} \subseteq \mathbb{X}$ satisfying the theorem is said to be a *weak equivalence*.

Proposition

If G is a logical group, the inclusion of a subgroup $H \subseteq G$ is a weak equivalence if and only if H is a *dense* subset of G.

Weak equivalences

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if and only if, for each open subgroupoid $\mathbb{U} \subseteq \mathbb{X}$, the map

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Corollary (W.)

Two logical groupoids $\mathbb{X}, \mathbb{Y}$ are Morita equivalent if and only if there exist embeddings

$$\mathbb{X} \subseteq \mathbb{W} \supseteq \mathbb{Y}$$

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Weak equivalences

Theorem (W.)

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if and only if, for each open subgroupoid $\mathbb{U} \subseteq \mathbb{X}$, the map

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A subgroupoid inclusion $\mathbb{Y} \subseteq \mathbb{X}$ satisfying the theorem is said to be a *weak equivalence*.

Proposition

If G is a logical group, the inclusion of a subgroup $H \subseteq G$ is a weak equivalence if and only if H is a *dense* subset of G.

Corollary (W.)

Let $\mathbb{T}_1, \mathbb{T}_2$ be theories with representing groupoids of models $\mathbb{X}, \mathbb{Y}$.

Then $\mathbb{T}_1, \mathbb{T}_2$ are Morita equivalent if and only if there exist embeddings

$$\mathbb{X} \subseteq \mathbb{W} \supseteq \mathbb{Y}$$

that are weak equivalences.

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Thank you for listening