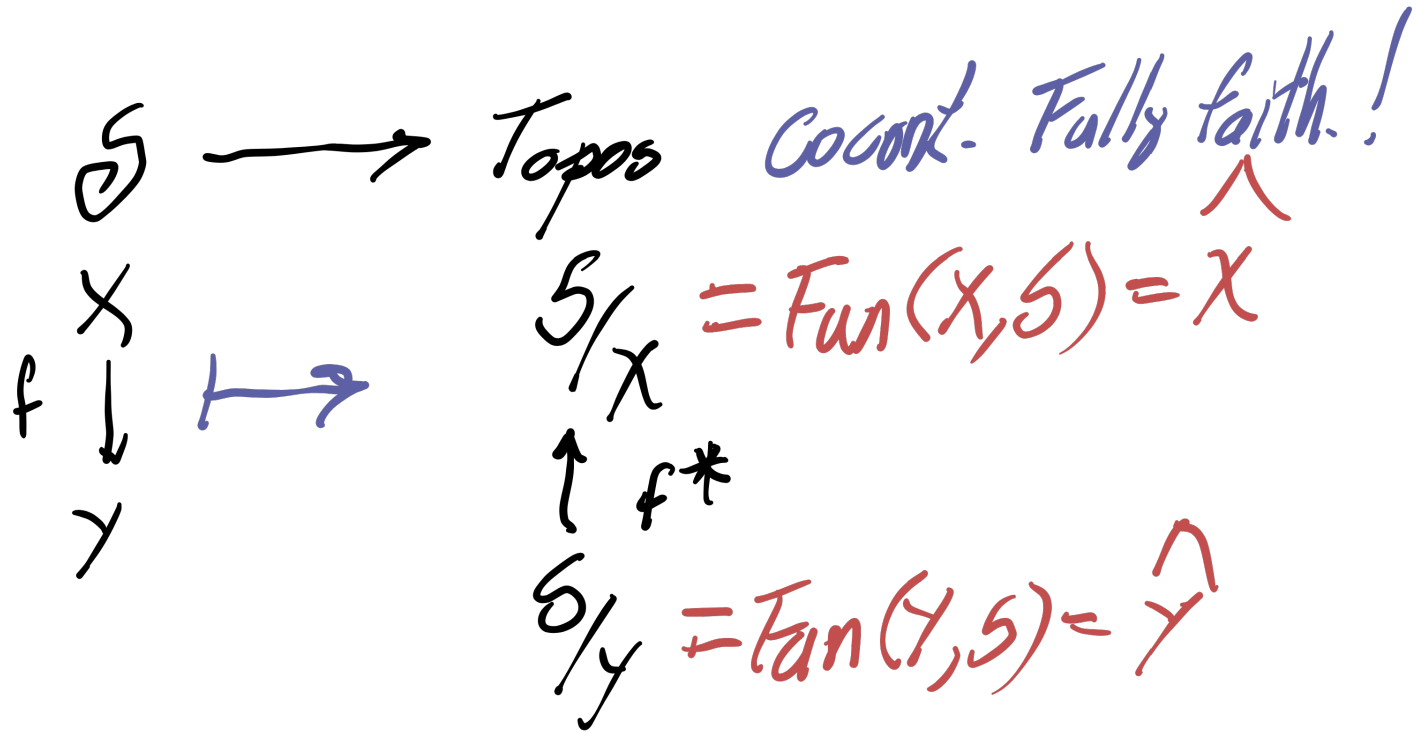


Some Homotopy Theories of Topoi

- Shape $\mathcal{T}ho(\mathcal{X})$
- Singular Homotopy Type $Sing(\mathcal{X})$
- Interval Equivalence

I) Shape and Singular Ho. Type



\exists Right adjoint and Pro-left adjoint

$$\mathcal{S}/\text{Sing}(\mathcal{X}) \longrightarrow \mathcal{X} \longrightarrow \mathcal{S}/\text{Pro}(\mathcal{X})$$

$$\text{Pro}(\mathcal{X}) \cong \Gamma_{\mathcal{X}}(\perp_{\mathcal{X}}) = \mathcal{X}(\perp_{\mathcal{X}}, \Gamma_{\mathcal{X}}^*(-)) \in \text{Pro}(\mathcal{S})$$

$\Gamma_{\mathcal{X}} \dashv \Gamma_{\mathcal{X}}^*$

lex
↑

op
↓

Ex. 1) X CW complex

$$\text{Sing}(\tilde{X}) = hX$$

$$\pi_{\infty}(\tilde{X}) = hX$$

Rem. X paracomp. Hausdorff

$$\pi_{\infty}(\tilde{X}) = \text{Ordinary Shape of } X$$

Ex. 2) $\mathcal{C} \in \text{Cat}$

$$\text{Sing}(\hat{\mathcal{C}}) = \mathcal{C}^{\cong}$$

$$\pi_0(\hat{\mathcal{C}}) = \mathcal{C} / \cong$$

||

$$\pi_0(\hat{\mathcal{C}}) = \text{Lan}_{\mathcal{C}^{\text{op}} \rightarrow 1} \mathbb{1}_{\mathcal{C}} = \text{colim}_{\mathcal{C}^{\text{op}}} \mathbb{1}_{\mathcal{C}}$$

II) Thom (Moerdijk) ξ small L -cat

$$[\tilde{x}, \hat{\xi}]_I \cong [\tilde{x}, \tilde{B\xi}]_I, \quad x \in CW$$

||

$$\text{Bun}_\xi(x) = [x, B\xi]$$

Homotopy
Classes

$$x \otimes \tilde{I} \rightarrow y \quad I = [0, 1]$$

$$[S/X, \hat{\theta}]_{\perp} \cong [S/X, S/\theta[\theta^{-1}]]_{\perp}$$

$$\frac{S}{\pi_0(\tilde{\theta})} \stackrel{||}{=} \frac{S}{\pi_0(\hat{\theta})}$$

$$[S/X, \hat{\theta}]_{\perp} \cong S(X, \theta[\theta^{-1}])$$

What's the relation between
 $\mathbb{T}_0(\mathcal{X})$ and $[-, \mathcal{X}]_I$?

Shape vs I -equivalences

III) Two Intervals

$$1) I = [0, 1] \mapsto \tilde{I} \mapsto \tilde{\Delta}_I^n$$

$$2) \Sigma = \text{Aptek}(\{0 \leq t \leq 1\}) \mapsto \tilde{\Delta}_\Sigma^n$$

Sierpiński
 n -simplex

\nearrow
 $\text{Aptek}([0, 1])$

$\mathcal{J} = I, \Sigma$ Realizations (Bunge, Moerdijk)

$$\dots \rightrightarrows \mathcal{X}^{\tilde{\Delta}_{\mathcal{J}}^2} \rightrightarrows \mathcal{X}^{\tilde{\Delta}_{\mathcal{J}}^1} \rightrightarrows \mathcal{X} \rightarrow \text{Sing}^{\mathcal{J}}(\mathcal{X})$$

$$\mathcal{X} \rightarrow \text{Sing}^{\mathcal{J}}(\mathcal{X}) \rightarrow \mathcal{S} / \mathcal{T}_{\text{ho}}(\mathcal{X}) \quad \text{admit in TopS}$$

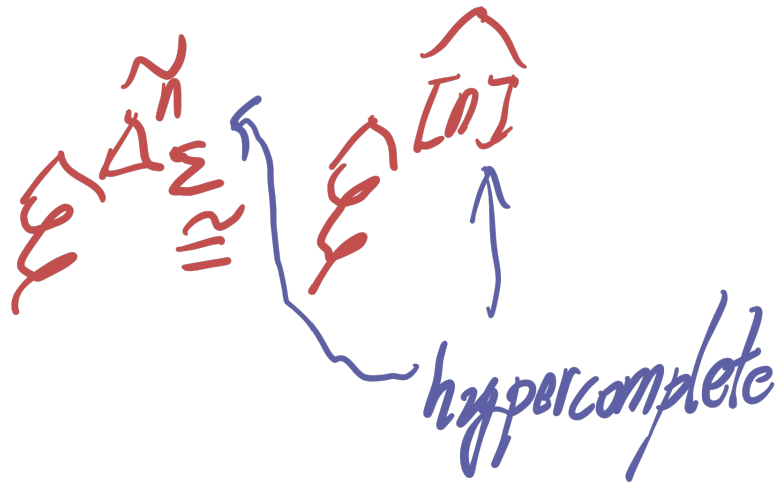
\uparrow
 finer invariant

Ex) $\hat{\mathbb{C}}$

$$\text{Sing}^{\Sigma}(\hat{\mathbb{C}}) = \text{colim}_{n \in \Delta^{op}} \mathbb{C} \xrightarrow{\sim} \Delta_{\Sigma}^{\sim}$$

$$\mathbb{C} \xrightarrow{\sim} \hat{[n]} \cong \mathbb{C} \xrightarrow{\sim} \hat{[n]}$$

(Lurie's SAG)



Therefore

In Topos

In CAT

$$\text{Sing}^{\Sigma}(\mathcal{C}) \cong \text{colim}_{n \in \Delta^{\text{op}}} \mathcal{C}^{[n]} \cong \lim_{n \in \Delta} \mathcal{C}^{[n]}$$

$$\cong \text{colim}_{n \in \Delta^{\text{op}}} \mathcal{C}^{[n]} \cong \mathcal{C}[\mathcal{C}^{-1}] = \mathcal{C} / \mathcal{C}[\mathcal{C}^{-1}] = \mathcal{C} / \text{Tho}(\mathcal{C})$$

$\mathcal{C}^{[n]} \cong \underbrace{\mathcal{C} \times \mathcal{C}_0 \cdots \times \mathcal{C}_0}_{n \text{ times}}$

Localizations $\mathcal{A} \hookrightarrow \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$

$\hat{\mathcal{C}}_{\mathcal{A}}$ \mathcal{A} -local (or \mathcal{A} -invariant) $\hookrightarrow \hat{\mathcal{C}}$

\mathcal{A} -equiv := saturation of \mathcal{A} or $f \in \hat{\mathcal{C}}_{\perp}$ s.t.

$\hat{\mathcal{C}}(f, -) \Big|_{\hat{\mathcal{C}}_{\mathcal{A}}}$ is iso

Our case, $\mathcal{C} := \text{Topos}$, $\mathcal{A}_J := \{ \mathcal{X} \otimes \tilde{\Delta}_J^n \}_{n \in \Delta}$ ($J = I, \Sigma$)

In summary,

$$\hat{\mathcal{C}} \rightarrow \text{Sing}^I(\hat{\mathcal{C}}) \rightarrow \text{Sing}^\Sigma(\hat{\mathcal{C}}) \xrightarrow{\sim} \mathcal{S} / \pi_0(\hat{\mathcal{C}})$$

Σ -equiv.
I-equiv.

$J = I, \Sigma$

J -equiv := \mathcal{A}_J -equiv.

saturation of $\{ \mathcal{X} \otimes \tilde{\Delta}_J^n \}_{n \in \Delta}$

Σ -equiv. \subset I-equiv.

$$\begin{array}{ccc} I & \longrightarrow & \Sigma \\ \uparrow & & \\ & & \text{I-equiv} \end{array}$$

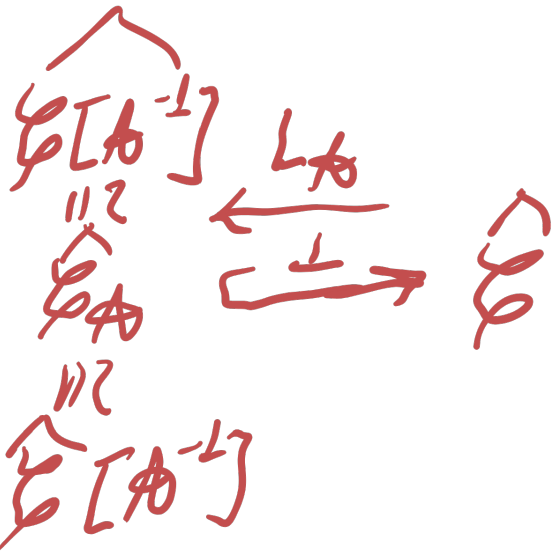
Prop (Hoyois)

- Let $\mathcal{A} \hookrightarrow \mathcal{C}$ closed under

base change

- $F \in \mathcal{C}$

- $\mathcal{A}_X \hookrightarrow \mathcal{C}/_X$ closure of $\mathcal{A}/_X$
under comp.
with $f \in \mathcal{A}$



Then $L_{\mathcal{A}}(F)(X) \cong \operatorname{colim}_{\mathcal{A}_X^{\text{op}}} F \circ (i_X : \mathcal{A}_X \hookrightarrow \mathcal{C}/_X \rightarrow \mathcal{C})^{\text{op}}$

Thm (Cisinski HA) Left Calc. of Fractions

$$W \hookrightarrow \mathcal{C}, Y \in \mathcal{C}, W(Y) \rightarrow \mathcal{C}/Y.$$

If $F(-) := \operatorname{colim} \mathcal{C}(-, Z)$ is W -local

$$\begin{array}{c} Y \\ \downarrow \\ Z \end{array} \in W(Y)_0$$

Then $F(-) \cong \mathcal{C}[W^{-1}](-, Y)$

$$\begin{array}{c} (-) \nearrow Z \leftarrow Y \\ \in W(Y) \end{array}$$

Therefore

$$W := \mathbb{A}_I, \quad W(\mathcal{Y}) := \{ \mathcal{Y}^{\tilde{\Delta}_I^n} \}_{n \in \Delta^{\text{op}}}$$

$$\text{colim}_{n \in \Delta^{\text{op}}} \text{Topos}(-, \mathcal{Y}^{\tilde{\Delta}_I^n}) \cong \text{colim}_{n \in \Delta^{\text{op}}} \text{Topos}((-) \otimes \tilde{\Delta}_I^n, \mathcal{Y})$$

↙ \mathbb{A}_I -local by Hoyo's Prop.

Hence, by left calc. of fractions,

$$\text{Topos}[\mathbb{A}_I^{-1}](\mathcal{X}, \mathcal{Y}) \cong \text{colim}_{n \in \Delta^{\text{op}}} \text{Topos}(\mathcal{X} \otimes \tilde{\Delta}_I^n, \mathcal{Y})$$

III) Conclusion

$$y := \hat{\varphi}, \mathcal{S}/\mathcal{B}[\varphi^{-1}], \tilde{\mathcal{B}}\varphi$$

$$y := \mathcal{S}/\mathcal{B}[\varphi^{-1}]$$

$$\mathcal{X} := \tilde{\mathcal{X}} \quad \mathcal{X} \in \text{Top}$$

$$\cong \mathcal{S}(\mathcal{X}, \mathcal{B}[\varphi^{-1}])$$

$$\text{colim}_{n \in \Delta^{\text{op}}} \text{Top}_{\mathcal{S}}(\tilde{\mathcal{X}} \times \Delta_{\mathcal{I}}^n, y)$$

All isomorphic for any y as above

$$\tilde{\mathcal{X}} \otimes \Delta_{\mathcal{I}}^n \xrightarrow{\text{loc. comp.}}$$

$\tilde{B}\mathcal{P}$ case

$$\|N(\mathcal{P})\|_I \xrightarrow{\text{I-equiv.}} \|N(\mathcal{P})\|_\Sigma$$

$\underset{\sim}{\tilde{B}\mathcal{P}} \quad \underset{\sim}{\mathcal{S}/\mathcal{P}[\mathcal{P}^{-1}]}$

$$\begin{aligned} \dots & \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \text{colim}_{\chi_2} \tilde{\Delta}_J^2 \rightarrow \text{colim}_{\chi_1} \tilde{\Delta}_J^1 \rightarrow \text{colim}_{\chi_0} \tilde{\Delta}_J^0 \rightarrow \|X\|_J \\ & J = \Sigma, I \quad X \in \mathcal{S}^{\Delta^{op}} \end{aligned}$$

\uparrow colim

IV) Further directions

1) What's the largest collection in Topos s.t.

$$\mathcal{X} \begin{array}{l} \longrightarrow \text{Sing}^{\mathcal{J}}(\mathcal{X}) \\ \searrow \\ \mathcal{S}/\Pi_0(\mathcal{X}) \end{array}$$

2) Compute $\text{Sing}^{\mathcal{J}}(\mathcal{X})$ and $\Pi_0(\mathcal{X})$ through loc. ho. types

