Some Homotopy Theories of Toposi

- Shape $\Pi_\infty(\mathcal{X})$
- Singular Homotopy Type $\mathrm{Sing}(\mathcal{X})$
- Interval Equivalence

I) Shape and Singular Ho. Type

$$\mathcal{S} \longrightarrow \mathrm{Topos}$$

Cocont. Fully faith.!

$$\begin{array}{ccc} X & & \mathcal{S}_{/X} = \mathrm{Fun}(X,\mathcal{S}) = \hat{X} \\ f \big\downarrow & \longmapsto & \big\uparrow f^* \\ Y & & \mathcal{S}_{/Y} = \mathrm{Fun}(Y,\mathcal{S}) = \hat{Y} \end{array}$$

$\exists$ Right adjoint and Pro-left adjoint

$$\mathcal{S}_{/\mathrm{Sing}(\mathcal{X})} \longrightarrow \mathcal{X} \longrightarrow \mathcal{S}_{/\Pi_\infty(\mathcal{X})}$$

$$\widehat{\mathcal{S}^{op}}^{op}$$

$$\uparrow$$

lex

$$\Pi_\infty(\mathcal{X}) \simeq \Gamma_{\mathcal{X}!}(1_{\mathcal{X}}) := \mathcal{X}(1_{\mathcal{X}}, \Gamma_{\mathcal{X}}^*(-)] \in \mathrm{Pro}(\mathcal{S})$$

$$\Gamma_{\mathcal{X}!} \dashv \Gamma_{\mathcal{X}}^*$$

Ex. 1) X CW complex

$$\mathrm{Sing}(\tilde{X}) = hX$$

$$\Pi_\infty(\tilde{X}) = hX$$

Rem. X paracomp. Hausdorff

$\Pi_\infty(\tilde{X}) =$ Ordinary Shape of X

Ex. 2) $\hat{\mathcal{C}} \quad \mathcal{C} \in Cat$

$$\mathrm{Sing}(\hat{\mathcal{C}}) = \mathcal{C}^{\simeq}$$

$$\Pi_\infty(\hat{\mathcal{C}}) = \mathcal{C}[\mathcal{C}^{-1}]$$

$$\shortparallel$$

$$\Gamma_{\hat{\mathcal{C}}!}(L\mathcal{C}) = \mathrm{Lan}_{\mathcal{C}^{op} \to 1} 1_{\mathcal{C}} = \mathrm{colim}_{\mathcal{C}^{op}} 1_{\mathcal{C}}$$

II) Thm (Moerdijk) $\mathcal{C}$ small $\mathcal{L}$-cat

$$[\tilde{X}, \hat{\mathcal{C}}]_I \cong [\tilde{X}, \widetilde{B\mathcal{C}}]_I \,, \quad X \in CW$$

Homotopy classes

$$\mathrm{Bun}_{\mathcal{C}}(X) = [X, B\mathcal{C}]$$

$\mathcal{X} \otimes \tilde{I} \to \mathcal{Y} \quad I = [0,1]$

$$[S/X, \hat{\phi}]_I \simeq [S/X, S/\phi[\phi^{-1}]]_I$$

$$\begin{array}{c} \| \\ S/\pi_0(B\tilde{\phi}) = S/\pi_0(\hat{\phi}) \end{array}$$

$$[S/X, \hat{\phi}]_I \simeq S(X, \phi[\phi^{-1}])$$

What's the relation between $\Pi_\infty(\mathcal{X})$ and $[-,\mathcal{X}]_I$?

Shape vs I-equivalences

III) Two Intervals

1) $I = [0,1] \longmapsto \tilde{I} \longmapsto \tilde{\Delta}^n_I$

2) $\Sigma = \mathrm{Assoc}(\{0<1\}) \longmapsto \tilde{\Delta}^n_\Sigma$

$$\tilde{\Delta}^n_\Sigma = \mathrm{Assoc}([n])$$

Sierpiński
n-simplex

$J = I, E$ Realizations (Bunge, Moerdijk)

$$\cdots \;\; \mathcal{X}^{\tilde{\Delta}^2_J} \;\; \mathcal{X}^{\tilde{\Delta}^1_J} \rightrightarrows \mathcal{X} \longrightarrow \mathrm{Sing}^J(\mathcal{X})$$

$$\mathcal{X} \longrightarrow \mathrm{Sing}^J(\mathcal{X}) \longrightarrow S/\Pi_\infty(\mathcal{X})$$

finer invariant

colimit in Topos

Ex) $\hat{\mathcal{C}}$

$$\mathrm{Sing}^{\Sigma}(\hat{\mathcal{C}}) = \operatorname*{colim}_{n\in\Delta^{op}} \hat{\mathcal{C}}^{\widetilde{\Delta}^n_\Sigma}$$

$$\hat{\mathcal{C}}^{\widehat{[n]}} \cong \widehat{\mathcal{C}^{[n]}}$$

(Lurie's SAG)

$$\hat{\mathcal{C}}^{\widetilde{\Delta}^n_\Sigma} \cong \hat{\mathcal{C}}^{\widehat{[n]}}$$

hypercomplete

Therefore

In Topos   In CAT

$$\mathrm{Sing}^{\Sigma}(\hat{\varphi}) \simeq \operatorname*{colim}_{n\in\Delta^{op}} \widehat{\varphi^{[n]}} \simeq \lim_{n\in\Delta} \widehat{\varphi^{[n]}} \simeq$$

$$\simeq \widehat{\operatorname*{colim}_{n\in\Delta^{op}} \varphi^{[n]}} \simeq \widehat{\varphi[\varphi^{-1}]} = \mathcal{S}/_{\varphi[\varphi^{-1}]} = \mathcal{S}/_{\Pi_\infty(\hat{\varphi})}$$

$$\varphi^{[n]} \simeq \overbrace{\varphi_1 \times_{\varphi_0} \cdots \times_{\varphi_0} \varphi_1}^{n \text{ times}}$$

Localizations $\mathcal{A} \hookrightarrow \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$

$\hat{\mathcal{C}}_{\mathcal{A}}$ $\mathcal{A}$-local (or $\mathcal{A}$-invariant) $\hookrightarrow \hat{\mathcal{C}}$

$\mathcal{A}$-equiv := saturation of $\mathcal{A}$ or $f \in \hat{\mathcal{C}}_1$ s.t.

$\hat{\mathcal{C}}(f,-)|_{\hat{\mathcal{C}}_{\mathcal{A}}}$ is iso

Our case, $\mathcal{C} :=$ Topos, $\mathcal{A}_J := \{ \mathcal{X} \otimes \tilde{\Delta}^n_J \}_{n \in \Delta}$ $(J = I, \Sigma)$

In summary,

$$\begin{tikzcd} \hat{\mathscr{C}} \arrow[r] \arrow[rr, bend right, "\Sigma\text{-equiv.}"'] & \mathrm{Sing}^{I}(\hat{\mathscr{C}}) \arrow[r] \arrow[r, bend right=60, "I\text{-equiv.}"'] & \mathrm{Sing}^{\Sigma}(\hat{\mathscr{C}}) \arrow[r, "\sim"] & \mathcal{S}/\Pi_\infty(\hat{\mathscr{C}}) \end{tikzcd}$$

$J = I, \Sigma$

J-equiv. $:= \mathcal{A}_J$-equiv.

$\parallel$

saturation of $\{\mathcal{X} \otimes \tilde{\Delta}^n_J\}_{n \in \Delta}$

Σ-equiv. $\subset I$-equiv.

$$I \xrightarrow[\uparrow\, I\text{-equiv}]{} \Sigma$$

Prop (Hoyois)

- Let $\mathcal{A} \hookrightarrow \mathcal{C}$ closed under base change
- $F \in \hat{\mathcal{C}}$
- $\mathcal{A}_X \hookrightarrow \mathcal{C}/_X$ closure of $\mathcal{A}/_X$ under comp. with $f \in \mathcal{A}$

$$\widehat{\mathcal{C}[\mathcal{A}^{-1}]} \cong \hat{\mathcal{C}}_{\mathcal{A}} \cong \hat{\mathcal{C}}[\mathcal{A}^{-1}] \qquad \widehat{\mathcal{C}[\mathcal{A}^{-1}]} \underset{\hookrightarrow}{\overset{L_{\mathcal{A}}}{\underset{\perp}{\leftarrow}}} \hat{\mathcal{C}}$$

Then $L_{\mathcal{A}}(F)(X) \cong \operatorname{colim}_{\mathcal{A}_X^{op}} F \circ (i_X : \mathcal{A}_X \hookrightarrow \mathcal{C}/_X \to \mathcal{C})^{op}$

Thm (Cisinski HA) Left Calc. of Fractions

$W \hookrightarrow \mathcal{C}$, $Y \in \mathcal{C}$, $W(Y) \to \mathcal{C}/Y$.

If $F(-) := \operatorname*{colim}_{\substack{Y \\ \downarrow \in W(Y)_0 \\ Z}} \mathcal{C}(-, Z)$ is W-local

Then $F(-) \cong \mathcal{C}[W^{-1}](-, Y)$

$$(-) \longrightarrow Z \xleftarrow{\ \in W(Y)\ } Y$$

Therefore

$$\mathcal{W} := \mathbb{A}_{\mathbb{I}}\,, \quad \mathcal{W}(\mathcal{Y}) := \{\mathcal{Y}^{\widetilde{\Delta}^n_{\mathbb{I}}}\}_{n\in\Delta^{op}}$$

$$\operatorname*{colim}_{n\in\Delta^{op}} \mathrm{Topos}(-,\mathcal{Y}^{\widetilde{\Delta}^n_{\mathbb{I}}}) \simeq \operatorname*{colim}_{n\in\Delta^{op}} \mathrm{Topos}((-)\otimes\widetilde{\Delta}^n_{\mathbb{I}},\mathcal{Y})$$

$\mathbb{A}_{\mathbb{I}}$-local by Hoyois' Prop.

Hence, by left calc. of fractions,

$$\mathrm{Topos}[\mathbb{A}_{\mathbb{I}}^{-1}](\mathcal{X},\mathcal{Y}) \simeq \operatorname*{colim}_{n\in\Delta^{op}} \mathrm{Topos}(\mathcal{X}\otimes\widetilde{\Delta}^n_{\mathbb{I}},\mathcal{Y})$$

III) Conclusion

$\mathcal{Y} := \hat{\mathcal{C}},\ \mathcal{S}/_{\mathcal{C}[\mathcal{C}^{-1}]},\ \widetilde{B\mathcal{C}}$ $\qquad$ $\mathcal{Y} := \mathcal{S}/_{\mathcal{C}[\mathcal{C}^{-1}]}$

$\mathcal{X} := \tilde{X} \quad X \in \mathrm{Top}$

$$\underset{n \in \Delta^{op}}{\mathrm{colim}}\ \mathrm{Topos}(\widetilde{X \times \Delta^n_I}, \mathcal{Y}) \overset{\sim}{=} \mathcal{S}(hX, \mathcal{C}[\mathcal{C}^{-1}])$$

$$\begin{matrix} \mathrm{Topos}(\widetilde{X \times \Delta^n_I}, \mathcal{Y}) \\ \text{\rotatebox{90}{$\cong$}} \\ \tilde{X} \otimes \tilde{\Delta}^n_I \end{matrix}$$

loc. comp.

All isomorphic for any $\mathcal{Y}$ as above

$\widetilde{B\mathscr{C}}$ case

$$\begin{array}{ccc} \|N(\mathscr{C})\|_I & \xrightarrow{I\text{-equiv.}\,!} & \|N(\mathscr{C})\|_\Sigma \\ \wr\| & & \wr\| \\ \widetilde{B\mathscr{C}} & & \mathcal{S}/\mathscr{C}[\mathscr{C}^{-1}] \end{array}$$

$$\cdots \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \operatorname{colim}_{X_2} \tilde{\Delta}^2_J \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \operatorname{colim}_{X_1} \tilde{\Delta}^1_J \;\substack{\rightarrow\\\rightarrow}\; \operatorname{colim}_{X_0} \tilde{\Delta}^0_J \longrightarrow \|X\|_J$$

colim

$J = \Sigma, I \qquad X \in \mathcal{S}^{\Delta^{op}}$

IV) Further directions

1) What's the largest collection in Topos s.t.

$$
\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[dr] & \mathrm{Sing}^{J}(\mathcal{X}) \arrow[d, phantom, sloped, "\simeq"] \\
& \mathcal{S}_{/\Pi_\infty(\mathcal{X})}
\end{tikzcd}
$$

2) Compute $\mathrm{Sing}^{J}(\mathcal{X})$ and $\Pi_\infty(\mathcal{X})$ through loc. ho. types