(Co)Fibrations, (pseudo)distributive laws and (quasi)toposes

Toposes in Mondovì

Igor Baković

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Table of contest

(Co)lax monads, lax algebras and associated (co)fibrations

Definition

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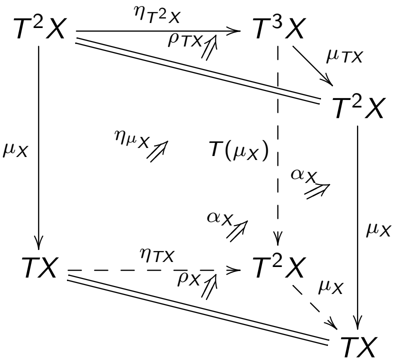
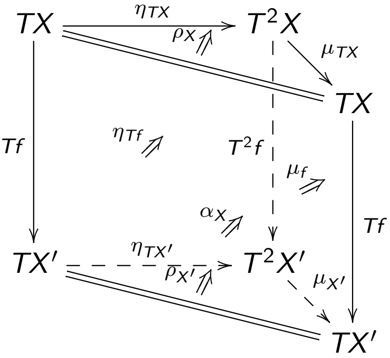
4) families $\lambda_X \colon \mu_X T(\eta_X) \Rightarrow 1_{TX}$, $\rho_X \colon 1_{TX} \Rightarrow \mu_X \circ \eta_{TX}$ and $\alpha_X \colon T\mu_X \circ \mu_X \Rightarrow \mu_{TX} \circ \mu_X$ of 2-cells in $\mathscr{K}$

such that the following axioms are satisfied:

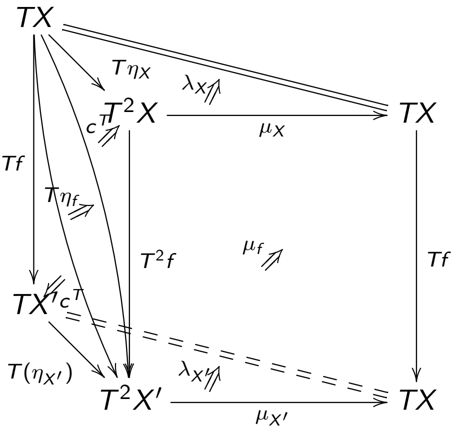
1) $(\alpha_X \circ \eta_{T^2X)})(\mu_X \circ \eta_{\mu_X})(\rho_X \circ \mu_X) = \mu_X \circ \rho_{TX} \colon \mu_X \Rightarrow \mu_X \circ \mu_{TX} \circ \eta_{T^2X}$



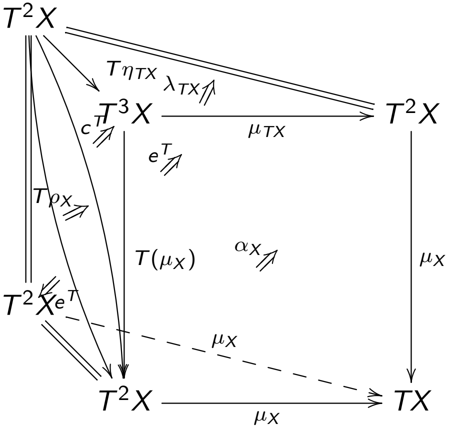
2) $(\mu_f \circ \eta_{TX)}(\mu_{X'} \circ \eta_{Tf})(\rho_{X'} \circ Tf) = Tf \circ \rho_X : Tf \Rightarrow Tf \circ \mu_X \circ \eta_{TX'}$, $\forall f : X \to X'$



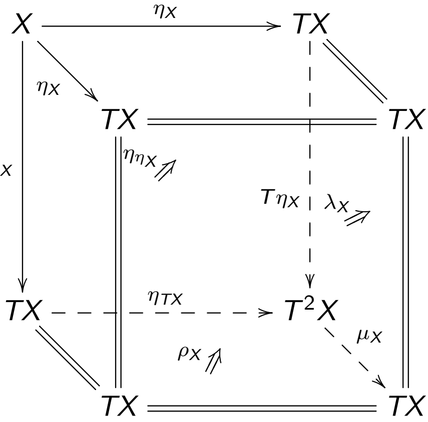
3) $(Tf \circ \lambda_X)(\mu_f \circ T\eta_X)(\mu_{X'} \circ c^T)(\mu_{X'} \circ T\eta_f) =$
$(\lambda_{X'} \circ Tf)(\mu_{X'} \circ c^T) \colon \mu_{X'} \circ T(\eta_{X'} \circ f) \Rightarrow Tf, \forall f \colon X \to X'$



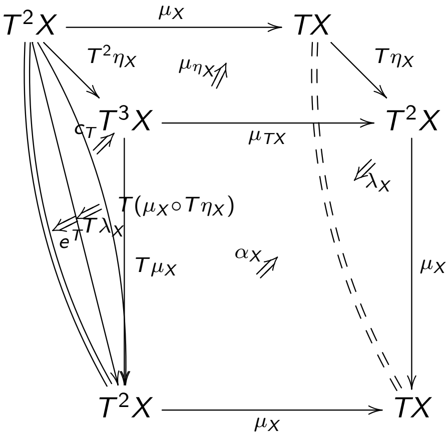
4) $(\mu_X \circ \lambda_{TX})(\alpha_X \circ T\eta_{TX})(\mu_X \circ c^T)(\mu_X \circ T\rho_X) = \mu_X \circ e^T \colon \mu_X \circ T1_{TX} \Rightarrow \mu_X \circ 1_{T^2X}$



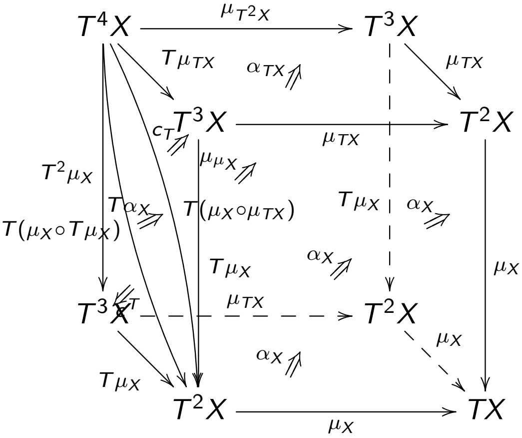
5) $(\lambda_X \circ \eta_X)(\mu_X \circ \eta_{\eta_X})(\rho_X \circ \eta_X) = 1_{\eta_X}$



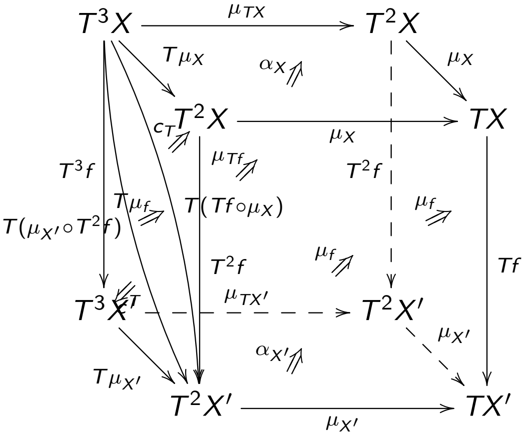
6) $(\lambda_X \circ \mu_X)(\mu_X \circ \mu_{\eta_X})(\alpha_X \circ T^2\eta_X)(\mu_X \circ c^T) = (\mu_X \circ e^T)(\mu_X \circ T\lambda_X)\colon \mu_X \circ T(\mu_X \circ T\eta_X) \Rightarrow \mu_X$



7) $(\alpha_X \circ \mu_{T^2X})(\mu_X \circ \mu_{\mu_X})(\alpha_X \circ T^2\mu_X)(\mu_X \circ c^T) = (\mu_X \circ \alpha_{TX})(\alpha_X \circ T\mu_{TX})(\mu_X \circ c^T)(\mu_X \circ T\alpha_X) \colon \mu_X \circ T(\mu_X \circ T\mu_X) \Rightarrow \mu_X \circ \mu_{TX} \circ \mu_{T^2X}$



8) $(\mu_f \circ \mu_{TX})(\mu_{X'} \circ \mu_{Tf})(\alpha_X \circ T^3 f)(\mu_{X'} \circ c^T) = (Tf \circ \alpha_X)(\mu_f \circ T\mu_X)(\mu_{X'} \circ c^T)(\mu_{X'} \circ T\mu_f) \colon \mu_{X'} \circ T(\mu_{X'} \circ T^2 f) \Rightarrow Tf \circ \mu_X \circ \mu_{TX}, \forall f : X \to X'$



Definition

Let $T\colon \mathscr{K} \to \mathscr{K}$ be an underlying colax functor of a colax monad $(T, \eta, \mu, \lambda, \rho, \alpha)$ on the 2-category $\mathscr{K}$. A lax T-algebra $(X, \xi, \iota_\xi, \kappa_\xi)$ consists of:

1) an object X of $\mathscr{K}$

such that the following axioms are satisfied:

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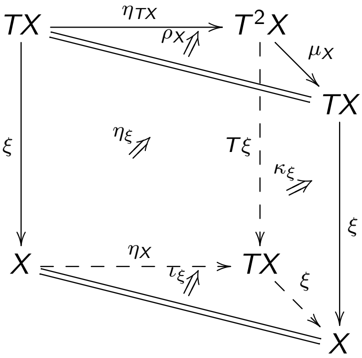
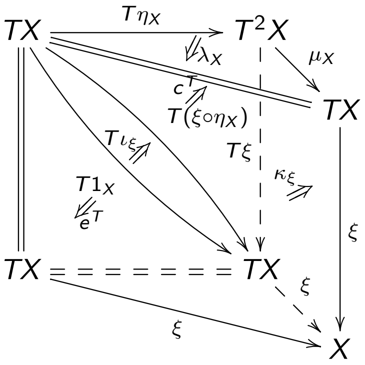
4) a 2-cell $\kappa_\xi\colon \xi \circ T\xi \Rightarrow \xi \circ \mu_X$ of $\mathscr{K}$

such that the following axioms are satisfied:

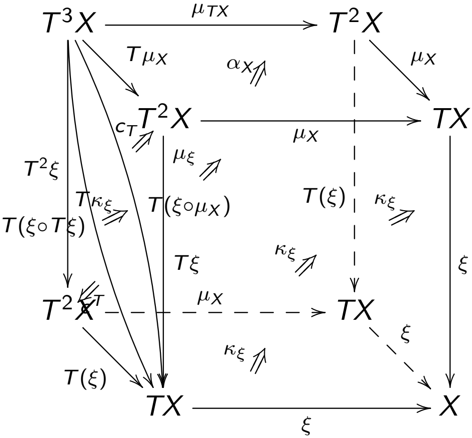
1) $(\kappa_\xi \circ \eta_{TX})(\xi \circ \eta_\xi)(\iota_\xi \circ \xi) = \xi \circ \rho_X : \xi \Rightarrow \xi \circ \mu_X \circ \eta_{TX}$



2) $(\xi \circ \lambda_X)(\kappa_\xi \circ T\eta_X)(\xi \circ c^T)(\xi \circ T\iota_\xi) = \xi \circ e^T \colon \xi \circ T1_X \Rightarrow \xi \circ 1_{TX}$



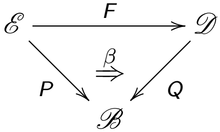
3) $(\kappa_\xi \circ \mu_{TX})(\xi \circ \mu_\xi)(\kappa_\xi \circ T^2\xi)(\xi \circ c^T) = (\xi\circ\alpha_X)(\kappa_\xi\circ T\mu_X)(\xi\circ c^T)(\xi\circ T\kappa_\xi)\colon \xi\circ T(\xi\circ T\xi) \Rightarrow \xi\circ\mu_X\circ\mu_{TX}$



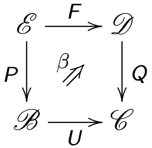
Extension of the definition of the associated split fibration

We consider functors as *generalized fibrations* (following Bénabou) in order to extend the definition of associated split fibration

1) from the 2-category $(\mathscr{C}at, \mathscr{B})$ whose 1-cells are triangles



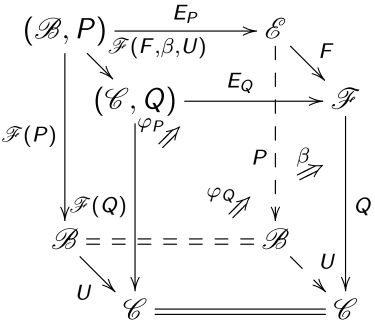
2) to the 2-category $\mathscr{C}at^2_c$ whose 1-cells are colax squares



3) ultimately to the double category $\mathbb{C}at^2$ whose horizontal (vertical) cells are (co)lax squares.

Associated split fibration 2-monad

Consider the following square



$\mathscr{F}(P)\colon (\mathscr{B}, P) \to \mathscr{B}$ and $E_P\colon (\mathscr{B}, P) \to \mathscr{E}$ send any object (B, p, E) in $(\mathscr{B}, P)$ (where $p\colon B \to P(E)$) to B and E respectively.

Associated split fibration 2-monad

From the universal property of comma squares there exists a unique functor $\mathscr{F}(F,\beta,U)\colon(\mathscr{B},P)\to(\mathscr{C},Q)$ which takes any object (B,p,E) in $(\mathscr{B},P)$ to $(U(B),\beta_E U(p),F(E))$ and any morphism $(u,e)\colon(B,p,E)\to(B',p',E')$ to the morphism $\mathscr{F}(F,\beta,U)(u,e):=(U(u),F(e))$ represented by a diagram

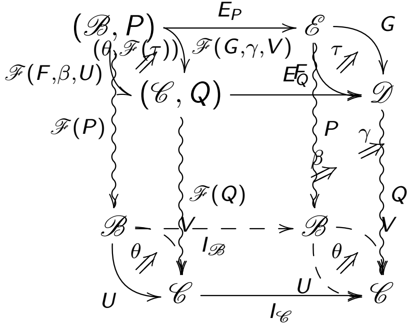
$$
\begin{array}{ccc}
U(B) & \xrightarrow{U(u)} & U(B') \\
{\scriptstyle U(p)}\downarrow & & \downarrow{\scriptstyle U(p')} \\
UP(E) & \xrightarrow[UP(e)]{} & UP(E') \\
{\scriptstyle \beta_E}\downarrow & & \downarrow{\scriptstyle \beta_{E'}} \\
QF(E) & \xrightarrow[QF(e)]{} & QF(E')
\end{array}
$$

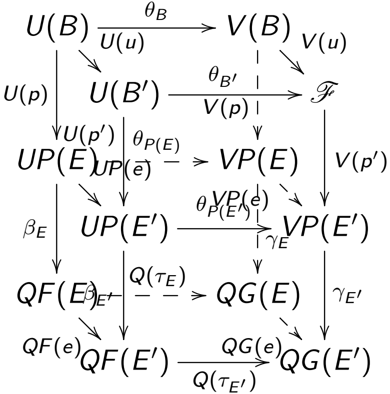
Theorem

There exists a colax idempotent 2-monad whose underlying 2-functor

$$\mathscr{F} : \mathscr{C}at^2_c \to \mathscr{C}at^2_c$$

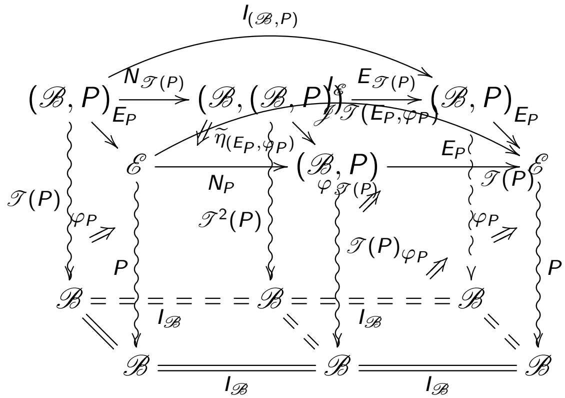
is given by the above construction.





Functors $\mathscr{F}(F,\beta,U)$ and $\mathscr{F}(G,\gamma,V)$ take an object (B,p,E) to $\mathscr{F}(F,\beta,U)(B,p,E) := (U(B), \beta_E U(p), F(E))$ and $\mathscr{F}(G,\gamma,V)(B,p,Q) := (V(B), \gamma_E V(p), G(E))$ respectively.

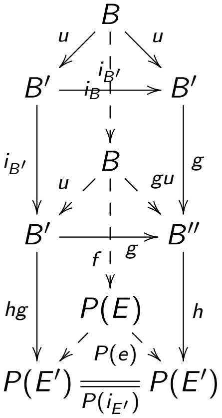
Components of the unit N_P and multiplication M_P of $\mathscr{F}$

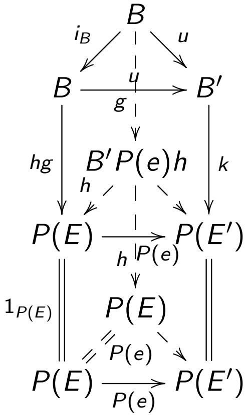


$$N_P(E) = (P(E), 1_{P(E)}, E)$$

$$M_P = \mathscr{F}(E_P, \varphi_P), \qquad M_P(B, f, B', g, E) = (B, gf, E)$$

Universal properties of local units and counits of $\mathscr{F}$

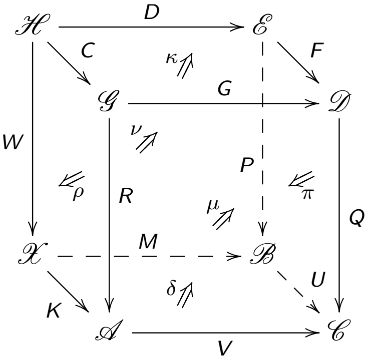


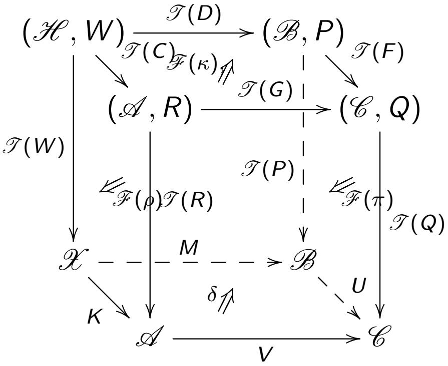


are counit and unit of the fully faithful adjoint triple

$$N_{\mathscr{T}(P)} \dashv \mathscr{T}(E_P) \dashv \mathscr{T}(N_P)$$

The associated split fibration $\mathscr{F}$ double monad





The definition requires the existence of pulbacks in base categories! Its domain is a double 2-category $(\mathbb{C}at, \mathbb{C}art)$ where $\mathbb{C}art$ is an (enhanced) 2-category of categories with pullbacks.

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{array}{c} B \\ \Big\downarrow{\scriptstyle p} \\ P(E) \end{array}$$

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$$\begin{array}{c} U(B) \\ \big\downarrow {\scriptstyle U(p)} \\ UP(E) \end{array}$$

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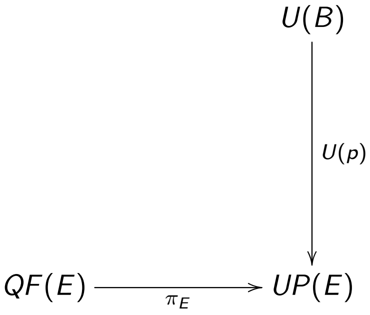

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{array}{ccc}
QF(E) \times_{UP(E)} U(B) & \xrightarrow{\mathscr{F}(\pi)_{(B,p,E)}} & U(B) \\
{\scriptstyle pr_1 = \mathscr{F}(F)(B,p,E)} \Big\downarrow & & \Big\downarrow {\scriptstyle U(p)} \\
QF(E) & \xrightarrow[\pi_E]{} & UP(E)
\end{array}
$$

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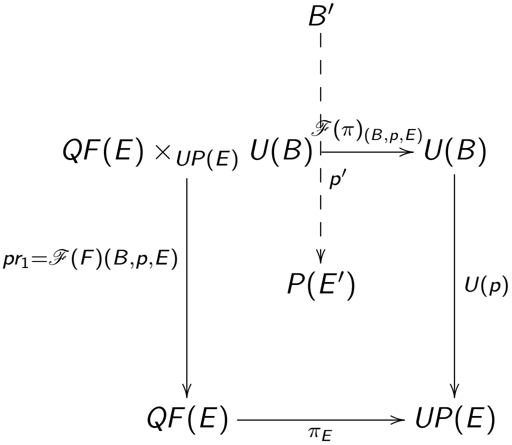

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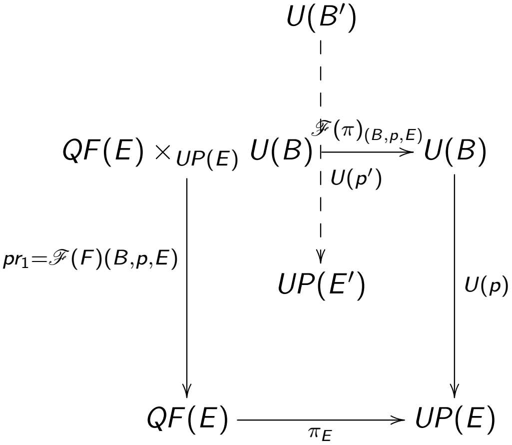

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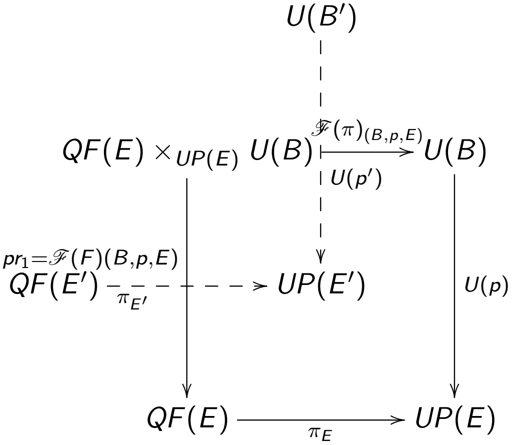

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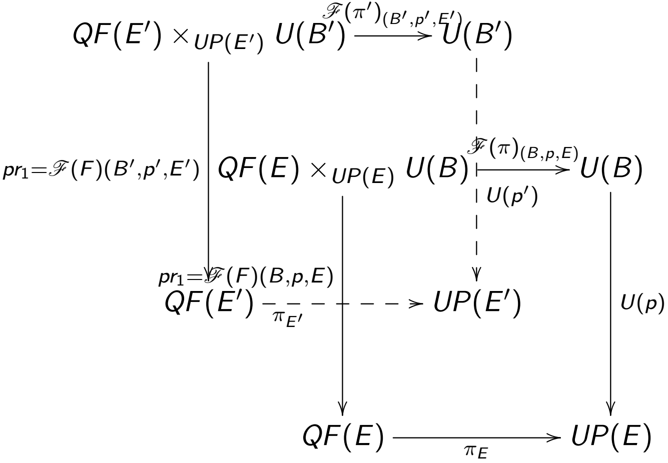

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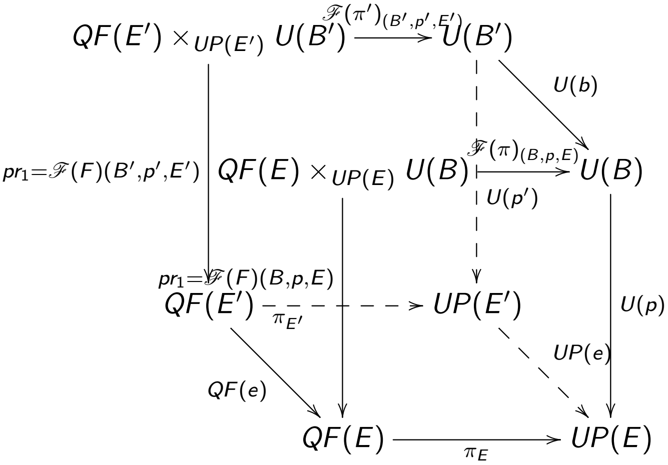
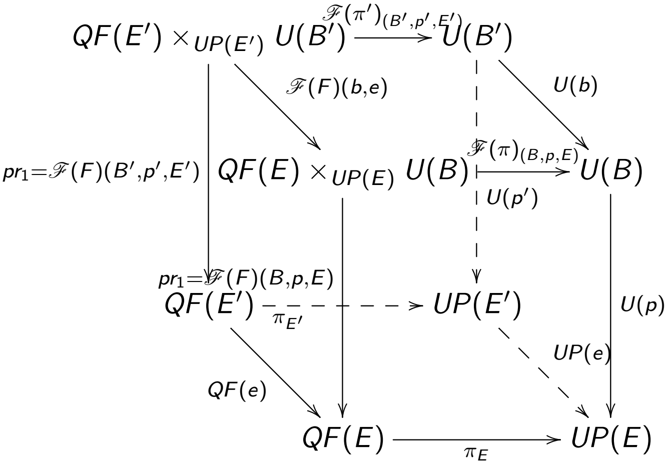


Definition of the associated split fibration $\mathscr{F}$ on lax squares



The associated split cofibration $\mathscr{F}^\circ$ 2-monad

1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

$$\mathscr{F}^\circ(P) := (\mathscr{F}(P^{op}))^{op}$$

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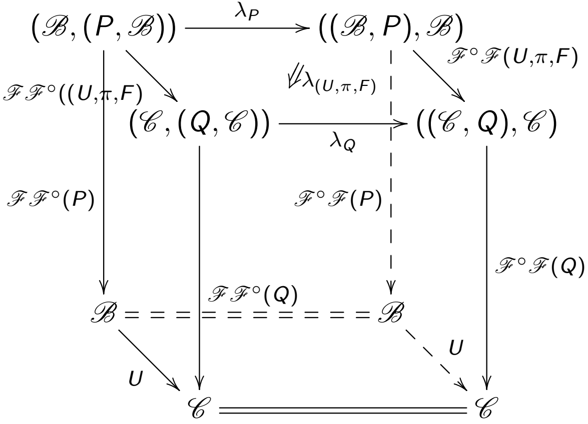
1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

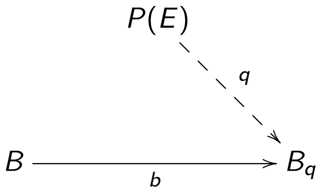
$$\mathscr{F}^\circ(P) := (\mathscr{F}(P^{op}))^{op}$$

2. It requires no conditions on lax squares
3. It requires the existence of pushouts in base categories for colax squares

Pseudo-distributive law between fibrations and cofibrations



The component of a lax-natural transformation λ



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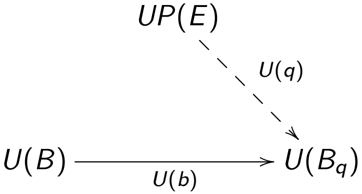

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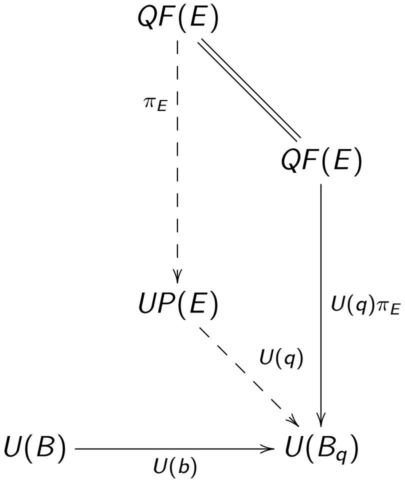

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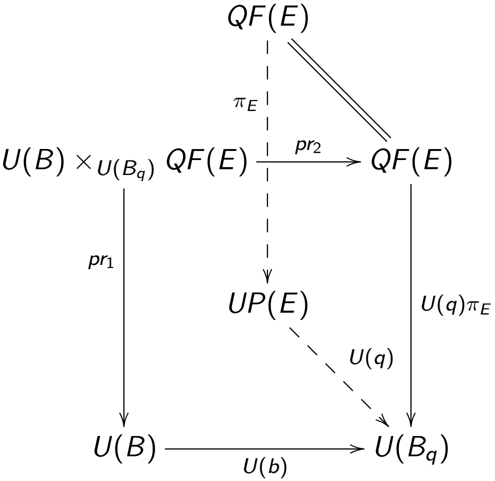

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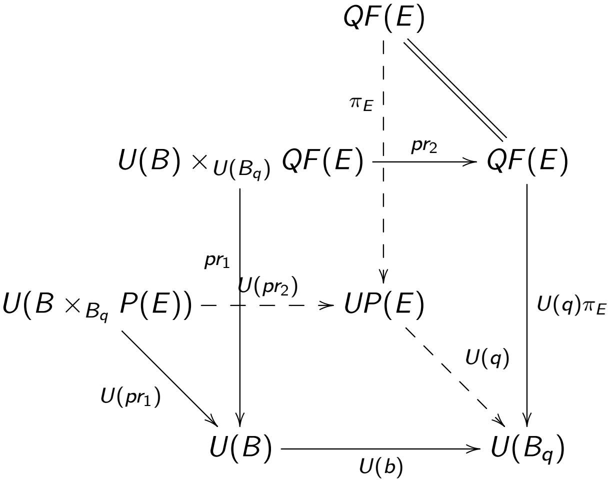

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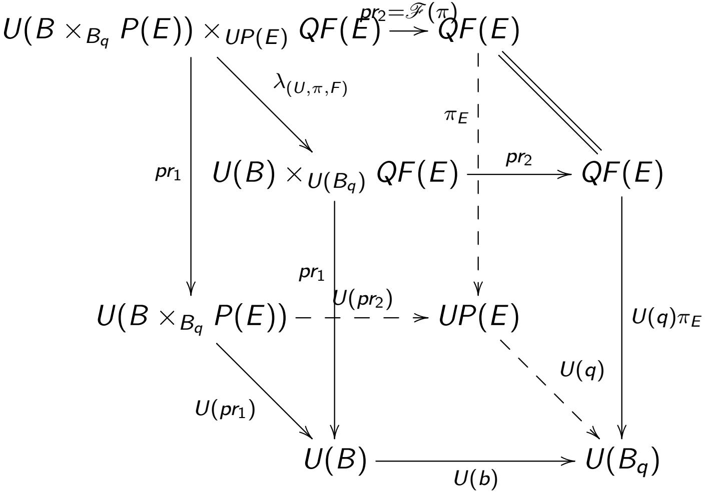

The associated Beck-Chevalley fibration

- The associated Beck-Chevalley fibrations are pseudoalgebras for the pseudo-distributive law

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- A natural candidate for the domain of its underlying 2-functor

$$\mathscr{F}\mathscr{F}^{\circ}\colon (\mathscr{C}at, \mathscr{QT}op) \to (\mathscr{C}at, \mathscr{QT}op)$$

is a double comma 2-category $(\mathscr{C}at, \mathscr{QT}op)$ where $\mathscr{QT}op$ is a 2-category of quasitoposes and geometric morphisms

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- A quasitopos is a finitely complete, finitely cocomplete, locally cartesian closed category $\mathscr{C}$ in which there exists an object Ω that classifies strong monomorphisms.

Admissibility

Admissible 1-cells

Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if its image $T(f)$ has a right adjoint μ_f. In the dual case of a colax idempotent 2-monad we say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if $T(f)$ has a left adjoint ν_f.

Admissible objects

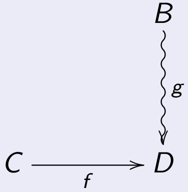
Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a (co)lax idempotent 2-monad on the 2-category $\mathscr{K}$ with a terminal object $\top$. We say that an object E of $\mathscr{K}$ is admissible if the unique 1-cell $!_E\colon E \to \top$ is admissible.

Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that (T, η, μ) is admissible if the following bicomma object condition holds:

1) the 2-category $\mathscr{K}$ has bicomma objects $f \downarrow g$ of diagrams



where 1-cells p and q are admissible.

2) the canonical 2-cell $T(q)\mu_p \Rightarrow \mu_f T(g)$ is a 2-isomorphism.

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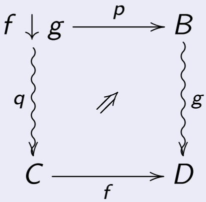

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The admissibility of associated split (co)fibrations

Theorem

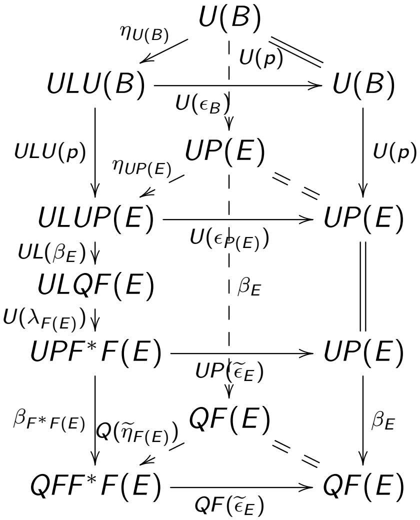
The associated split fibration 2-monad is admissible.

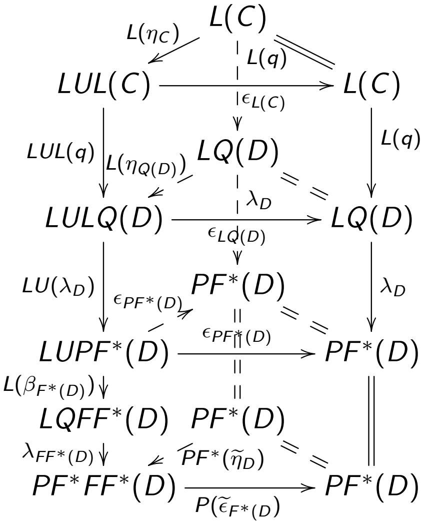
The two triangle identities

$$
\begin{array}{ccc}
 & \mathscr{F}(F,\beta,U) & \\
 {\scriptstyle \widetilde{\eta}\mathscr{F}(F,\beta,U)} \swarrow & & \| \\
\mathscr{F}(F,\beta,U)\mathscr{F}(F^*,\lambda,L)\mathscr{F}(F,\beta,U) & \xrightarrow[\mathscr{F}(F,\beta,U)\widetilde{\epsilon}]{} & \mathscr{F}(F,\beta,U)
\end{array}
$$

$$
\begin{array}{ccc}
 & \mathscr{F}(F^*,\lambda,L) & \\
 {\scriptstyle \mathscr{F}(F^*,\lambda,L)\widetilde{\eta}} \swarrow & & \| \\
\mathscr{F}(F^*,\lambda,L)\mathscr{F}(F,\beta,U)\mathscr{F}(F^*,\lambda,L) & \xrightarrow[\widetilde{\epsilon}\mathscr{F}(F^*,\lambda,L)]{} & \mathscr{F}(F^*,\lambda,L)
\end{array}
$$

are represented by the following diagrams


$U(B)$
$\eta_{U(B)}$
$U(p)$
$ULU(B)$
$U(B)$
$U(\epsilon_B)$
$ULU(p)$
$UP(E)$
$U(p)$
$\eta_{UP(E)}$
$ULUP(E)$
$UP(E)$
$U(\epsilon_{P(E)})$
$UL(\beta_E)$
$ULQF(E)$
β_E
$U(\lambda_{F(E)})$
$UPF^*F(E)$
$UP(E)$
$UP(\widetilde{\epsilon}_E)$
$\beta_{F^*F(E)}$
$Q(\widetilde{\eta}_{F(E)})$
$QF(E)$
β_E
$QFF^*F(E)$
$QF(E)$
$QF(\widetilde{\epsilon}_E)$


$L(C)$
$L(\eta_C)$
$L(q)$
$LUL(C)$
$L(C)$
$\epsilon_{L(C)}$
$L(q)$
$LUL(q)$
$L(\eta_{Q(D)})$
$LQ(D)$
λ_D
$LULQ(D)$
$LQ(D)$
$\epsilon_{LQ(D)}$
$LU(\lambda_D)$
$\epsilon_{PF^*(D)}$
$PF^*(D)$
λ_D
$\epsilon_{PF^*(D)}$
$LUPF^*(D)$
$PF^*(D)$
$L(\beta_{F^*(D)})$
$LQFF^*(D)$
$PF^*(D)$
$\lambda_{FF^*(D)}$
$PF^*(\widetilde{\eta}_D)$
$PF^*FF^*(D)$
$PF^*(D)$
$P(\widetilde{\epsilon}_{F^*(D)})$

Definition

A functor $U\colon \mathscr{A} \to \mathscr{B}$ is a local right adjoint if the restriction

$$U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$$

of U to the slice $(\mathscr{A}, A)$ category for each object A of $\mathscr{A}$ has a left adjoint

$$L_A\colon (\mathscr{B}, U(A)) \to (\mathscr{A}, A).$$

Equivalently, each fiber $U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$ of the diagram

$$\begin{array}{ccc} \mathscr{A}^2 & \xrightarrow{U^2} & \mathscr{B}^2 \\ {\scriptstyle cod}\downarrow & & \downarrow{\scriptstyle cod} \\ \mathscr{A} & \xrightarrow[U]{} & \mathscr{B} \end{array}$$

has a left adjoint.

Theorem

Right multiadjoints are admissible objects for the associated split fibration 2-monad.

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