Relative toposes for the working mathematician

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Plan of the course

- Motivation
- Preliminaries
  - Topos-theoretic background
  - Toposes as 'bridges'
  - Functors inducing morphisms of toposes
- New foundations for relative topos theory (j.w. with Riccardo Zanfa)
  - Relative presheaf toposes
    - Stacks and operations on them
    - The fundamental adjunction
    - The general presheaf-bundle adjunction
    - Some applications
  - Relative sheaf toposes
- Relative Diaconescu's equivalence (j.w. with Léo Bartoli)
- Existential fibred sites and their associated toposes
- Towards relative geometric logic (j.w. with Raffaele Lamagna)
- Future directions

Relativity techniques

- Broadly speaking, in Mathematics the relativization method consists in trying to state notions and results in terms of morphisms, rather than objects, of a given category, so that they can be 'relativized' to an arbitrary base object.
- One works in the new, relative universe as it were the 'classical' one, and then interprets the obtained results from the point of view of the original universe. This process is usually called *externalization*.
- Relativity techniques can be thought as general 'change of base techniques', allowing one to choose the universe relatively to which one works according to one's needs.
- The relativity method has been pioneered by Grothendieck, in particular for schemes, in his categorical refoundation of Algebraic Geometry, and has played a key role in his work.
- We aim for a similar set of tools for toposes, that is, for an efficient formalism for doing topos theory over an arbitrary base topos.

Potential for applications

- The relativity method provides an extremely high degree of technical flexibility, resulting from the possibility of 'encapsulating' part of the complexity of a situation in the base topos, so that the given notions acquire a simpler (e.g. a lower-degree) expression with respect to it.
- For example, a first-order theory becomes a propositional one when regarded relative to a suitable base topos.
- Several mathematicians have already started using relativity techniques for toposes:
  - Scholze and Clausen's condensed mathematics;
  - Tao and Jamneshan's topos-theoretic measure theory;
  - Tomasic's topos-theoretic difference algebra.
- It is therefore of fundamental importance to dispose of effective tools for working with relative toposes which are as general and flexible as possible, also from the computational point of view.

Topos theory over an arbitrary base topos

Our new foundations for relative topos theory are based on stacks (and, more generally, fibrations and indexed categories).

The approach of category theorists (Lawvere, Diaconescu, Johnstone, etc.) to this subject is chiefly based on the notions of internal category and of internal site.

The problem with these notions is that they are too rigid to naturally capture relative topos-theoretic phenomena, as well as for making computations and formalizing 'parametric reasoning'.

We have thus decided to resort to the more general and technically flexible notion of stack, developing the point of view originally introduced by J. Giraud in his paper *Classifying topos*.

Grothendieck topologies

Definition

Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Definition

A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
(iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.

Examples of Grothendieck topologies

- For any (small) category $\mathcal{C}$, the trivial topology on $\mathcal{C}$ is the Grothendieck topology in which the only sieve covering an object c is the maximal sieve M_c.
- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U\ .$$

- The Zariski topology on the opposite of the category $\mathbf{Rng}_{\text{f.g.}}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in $\mathbf{Rng}_{\text{f.g.}}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in $\mathbf{Rng}_{\text{f.g.}}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.
- Given a (first-order geometric) theory $\mathbb{T}$, one can naturally associate a site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ with it, called its *syntactic site*, which embodies essential aspects of the syntax and proof theory of $\mathbb{T}$.

Sites and presheaves

Definition

- A site (resp. small site) is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category (resp. a small category) and J is a Grothendieck topology on $\mathcal{C}$.
- A site $(\mathcal{C}, J)$ is said to be small-generated if $\mathcal{C}$ is locally small and has a small J-dense subcategory.

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d .$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S .$$

Sheaves on a site

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a *J*-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The J-sheaf condition can be expressed as the requirement that for every J-covering sieve S the canonical arrow

$$P(c) \to \prod_{f \in S} P(\mathrm{dom}(f))$$

given by $x \to (P(f)(x) \mid f \in S)$ should be the equalizer of the two arrows

$$\prod_{f \in S} P(\mathrm{dom}(f)) \to \prod_{\substack{\text{f,g, } f \in S \\ \mathrm{cod}(g)=\mathrm{dom}(f)}} P(\mathrm{dom}(g))$$

given by $(x_f \to (x_{f \circ g}))$ and $(x_f \to (P(g)(x_f)))$.

The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.

The notion of Grothendieck topos

Definition

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups:

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ is the category of sheaves $\mathbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.
- For any topological space X, $\mathbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\mathbf{Sh}(X)$ of sheaves on X.
- For any (topological) group G, the category $BG = \mathbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent to the category $\mathbf{Sh}(\mathbf{Cont}_{\text{t}}(G), J_{\text{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_{\text{t}}(G)$ on the non-empty transitive actions with respect to the atomic topology).

Basic properties of Grothendieck toposes

Grothendieck toposes satisfy all the categorical properties that one might hope for:

Theorem

Let $(\mathcal{C}, J)$ be a (small-generated) site. Then

- *the inclusion* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *has a left adjoint* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *(called the associated sheaf functor), which preserves finite limits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) limits, which are preserved by the inclusion functor* $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*; in particular, limits are computed pointwise and the terminal object* $1_{\mathbf{Sh}(\mathcal{C},J)}$ *of* $\mathbf{Sh}(\mathcal{C}, J)$ *is the functor* $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *sending each object* $c \in Ob(\mathcal{C})$ *to the singleton* $\{*\}$*.*
- *The associated sheaf functor* $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ *preserves colimits; in particular,* $\mathbf{Sh}(\mathcal{C}, J)$ *has all (small) colimits.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has exponentials, which are constructed as in the topos* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a subobject classifier.*
- *The category* $\mathbf{Sh}(\mathcal{C}, J)$ *has a separating set of objects (for instance, the one provided by the objects of the form* $l(c)$ *for* $c \in \mathcal{C}$*, where* l *is the canonical functor* $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$*).*

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism:

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

Grothendieck toposes, geometric morphisms and geometric transformations form a 2-category, called **Topos**.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.

- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\bigsqcup_{s \in S} 1_{\mathcal{E}}$.

- For any (small) site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Presentations of toposes

The most classical way for building toposes is through sites (indeed, a Grothendieck topos is, by definition, any category equivalent to the category of sheaves on a small-generated site).

Still, toposes can also be canonically associated with groups (or more generally topological or localic groupoids) or with (first-order geometric) theories or with non-commutative structures such as quantales or quantaloids, etc.

Every topos is associated with infinitely many presentations (in particular, with infinitely many sites of definition), which may belong to different areas of mathematics.

In this course we shall approach toposes from the geometric point of view of their site presentations.

Toposes as 'bridges'

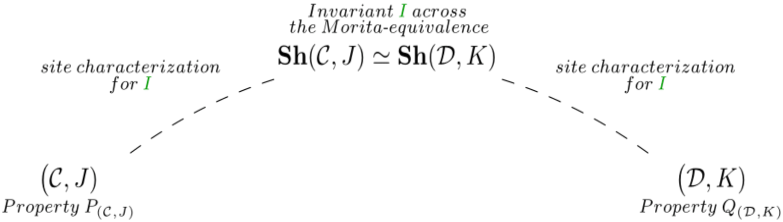
- One can exploit the duality between toposes and their presentations to build 'bridges' across different mathematical theories or contexts.
- More specifically, for any topos-theoretic invariant (i.e. notion which is invariant under categorical equivalence of toposes), one can try to construct 'bridges' by 'computing' it in terms of different presentations of a given topos.
- Provided that such 'unravelings' are technically feasible, this will result in correspondences between 'concrete' notions pertaining to the different presentations.
- The effectiveness of the 'bridge' technique actually relies on the natural structural relationship between a topos and its presentations.

Toposes as ‘bridges’

- These ‘bridges’ allow effective and often deep transfers of notions, ideas and results across the theories.
- Note that toposes disappear in the end, though they have been instrumental for performing the ‘translation’.
- In fact, ‘bridges’ have proved useful not only for connecting different theories with each other, but also for working inside a given mathematical theory and investigating it from a multiplicity of points of view.
- The level of mathematical depth of a ‘bridge’ may vary enormously from case to case, as it depends on the degree of sophistication of the invariant inducing it, in particular in relation to the given presentations, as well as on the complexity of the given equivalence of toposes. Still, even simple invariants applied to easy-to-establish equivalences can lead to surprising, deep results.

The ‘bridge’ technique

- Decks of ‘bridges’: Morita-equivalences (that is, equivalences between different presentations of a given topos, or more generally morphisms or other kinds of relations between toposes)
- Arches of ‘bridges’: Characterizations for topos-theoretic invariants in terms of different presentations of toposes



For example, this ‘bridge’ yields a logical equivalence between the ‘concrete’ properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the topo level.

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}, J')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that there is a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "l"'] & \mathcal{C}' \arrow[d, "l'"] \\ \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "u^*"] & \mathbf{Sh}(\mathcal{C}', J'); \end{tikzcd}$$

- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ which has the covering-lifting property (in the sense that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*
- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a geometric morphism $C_\pi : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*

Morphisms of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be small-generated sites. Then, given a functor $F : \mathcal{C} \to \mathcal{C}'$, the following conditions are equivalent:

(i) *F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$;*

(ii) *F satisfies the following properties:*

  (1) *F sends every J-covering family in $\mathcal{C}$ into a J'-covering family in $\mathcal{C}'$.*

  (2) *Every object c' of $\mathcal{C}'$ admits a J'-covering family*

$$c'_i \longrightarrow c', \quad i \in I,$$

  by objects c'_i of $\mathcal{C}'$ which have morphisms

$$c'_i \longrightarrow F(c_i)$$

  to the images under F of objects c_i of $\mathcal{C}$.

Morphisms of sites

(3) For any objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms of $\mathcal{C}'$

$$f'_1 : c' \longrightarrow F(c_1)\,, \quad f'_2 : c' \longrightarrow F(c_2)\,,$$

there exists a J'-covering family

$$g'_i : c'_i \longrightarrow c'\,, \quad i \in I\,,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f^i_1 : b_i \longrightarrow c_1\,, \quad f^i_2 : b_i \to c_2\,, \quad i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : c'_i \longrightarrow F(b_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_1"] \\ F(b_i) \arrow[r, "F(f^i_1)"] & F(c_1) \end{tikzcd} \qquad \begin{tikzcd} c'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & c' \arrow[d, "f'_2"] \\ F(b_i) \arrow[r, "F(f^i_2)"] & F(c_2) \end{tikzcd}$$

Morphisms of sites

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any arrow of $\mathcal{C}'$

$$f' : b' \longrightarrow F(c)$$

satisfying

$$F(f_1) \circ f' = F(f_2) \circ f' ,$$

there exist a J'-covering family

$$g'_i : b'_i \longrightarrow b' , \quad i \in I ,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c , \quad i \in I ,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i , \quad \forall\, i \in I ,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow F(b_i) , \quad i \in I ,$$

making commutative the following squares:

$$\begin{tikzcd} b'_i \arrow[r, "g'_i"] \arrow[d, "h'_i"'] & b' \arrow[d, "f'"] \\ F(b_i) \arrow[r, "F(h_i)"] & F(c) \end{tikzcd}$$

Kan extensions

The direct and image functors of geometric morphisms induced by morphisms or comorphisms of sites can be naturally described in terms of Kan extensions.

Recall that, given a functor $f : \mathcal{C} \to \mathcal{D}$,

- the right Kan extension $\mathrm{ran}_{f^{\mathrm{op}}}$ along f^{op}, which is right adjoint to the functor $f^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, is given by the following formula:

$$\mathrm{ran}_{f^{\mathrm{op}}}(F)(b) = \varprojlim_{\phi : fa \to b} F(a),$$

where the limit is taken over the opposite of the comma category $(f \downarrow b)$.

- the left Kan extension $\mathrm{lan}_{f^{\mathrm{op}}}$ along f^{op}, which is left adjoint to f^*, is given by the following formula:

$$\mathrm{lan}_{f^{\mathrm{op}}}(F)(b) = \varinjlim_{\phi : b \to fa} F(a),$$

where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

Geometric morphisms and Kan extensions

Proposition

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then*

- *the direct image $\mathbf{Sh}(F)_*$ of the geometric morphism*

$$\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

  induced by F is given by the restriction to sheaves of F^*;
- *the inverse image $\mathbf{Sh}(F)^*$ of $\mathbf{Sh}(F)$ is given by*

$$a_K \circ \mathrm{lan}_{F^{\mathrm{op}}} \circ i_J,$$

  where $\mathrm{lan}_{F^{\mathrm{op}}}$ is the left Kan extension and i_J is the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) *Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Then*

- *the direct image $(C_F)_*$ of the geometric morphism*

$$C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

  induced by F is given by the restriction to sheaves of the right Kan extension $\mathrm{ran}_{F^{\mathrm{op}}}$;
- *the inverse image $(C_F)^*$ of C_F is given by*

$$a_J \circ F^* \circ i_K,$$

  where i_K is the inclusion $\mathbf{Sh}(\mathcal{D}, K) \hookrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

Unifying morphisms and comorphisms of sites

We shall unify the notions of morphism and comorphisms of sites by interpreting them as two fundamentally different ways of describing morphisms of toposes which correspond to each other under a 'bridge'.

More specifically, morphisms of sites provide an 'algebraic' perspective on morphisms of toposes, while comorphisms of sites provide a 'geometric' perspective on them.

The key idea is to replace the given sites of definition with Morita-equivalent ones in such a way that the given morphism (resp. comorphism) of sites acquires a left (resp. right) adjoint, not necessarily in the classical categorial sense but in the weaker topos-theoretic sense of the associated comma categories having equivalent associated toposes.

From morphisms to comorphisms of sites

Theorem (O.C.)

Given a morphism $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ of small-generated sites, let

- $(1_{\mathcal{D}} \downarrow F)$ *be the 'comma category' whose objects are the triplets* $(d, c, \alpha : d \to F(c))$
- i_F *be the functor* $\mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ *sending any object c of $\mathcal{C}$ to the triplet* $(F(c), c, 1_{F(c)})$,
- $\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ *and* $\pi_{\mathcal{D}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ *the canonical projection functors, and*
- $\tilde{K}$ *be the Grothendieck topology on $(1_{\mathcal{D}} \downarrow F)$ whose covering sieves are those whose image under $\pi_{\mathcal{D}}$ is K-covering.*

Then:

(i) $\pi_{\mathcal{C}} \dashv i_F$, $\pi_{\mathcal{D}} \circ i_F = F$, *$i_F$ is a morphism of sites* $(\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ *and* $c_F := \pi_{\mathcal{C}}$ *is a comorphism of sites* $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$.

From morphisms to comorphisms of sites

(ii) $\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences

$$C_{\pi_{\mathcal{D}}} : \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \arrow[rr, shift left, "C_{\pi_{\mathcal{D}}}"] \arrow[rr, phantom, "\sim"] \arrow[dr, "C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_F)"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "\mathbf{Sh}(\pi_{\mathcal{D}})"] \arrow[dl, "\mathbf{Sh}(F)"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, f^* is a morphism of sites $(\mathcal{E}, J^{\text{can}}_{\mathcal{E}}) \to (\mathcal{F}, J^{\text{can}}_{\mathcal{F}})$ such that $f = \mathbf{Sh}(f^*)$. We thus obtain the following

Corollary (O.C.)

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

is a comorphism of sites $((1_{\mathcal{F}} \downarrow f^*), \widetilde{J^{\text{can}}_{\mathcal{F}}}) \to (\mathcal{E}, J^{\text{can}}_{\mathcal{E}})$ such that $f = C_{\pi_{\mathcal{E}}}$.

The canonical stack of a geometric morphism

The functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of f: from an indexed point of view, this stack sends any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.
We shall call the Grothendieck topology $\widetilde{J^{\text{can}}_{\mathcal{F}}}$ on $(1_{\mathcal{F}} \downarrow f^*)$ the relative topology of f and denote it by J_f.

By taking f to be the identity, and choosing a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$, we obtain the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ on $(\mathcal{C}, J)$, which sends any object c of $\mathcal{C}$ to the topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$. The above corollary thus specializes to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, \widetilde{J^{\text{can}}_{\mathbf{Sh}(\mathcal{C},J)}}),$$

which represents a 'thickening' of the usual representation of a Grothendieck topos as the topos of sheaves over itself with respect to the canonical topology.

From comorphisms to morphisms of sites

With a comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ we can associate the morphism of sites

$$m_F : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$$

sending an object c of $\mathcal{C}$ to the presheaf $\mathrm{Hom}_{\mathcal{C}}(F(-), c)$, where $\hat{K}$ is the extension of the Grothendieck topology K along the Yoneda embedding $y_{\mathcal{D}} : \mathcal{D} \to \hat{\mathcal{D}}$.

This morphism of sites induces a geometric morphism $\mathbf{Sh}(m_F)$ making the following triangle commute:

$$\begin{tikzcd}
\mathbf{Sh}(\hat{\mathcal{D}}, \hat{K}) \arrow[rr, shift left, "\mathbf{Sh}(y_{\mathcal{D}})"] \arrow[rr, phantom, "\sim"] \arrow[dr, "\mathbf{Sh}(m_F)"'] & & \mathbf{Sh}(\mathcal{D}, K) \arrow[ll, shift left, "C_{y_{\mathcal{D}}}"] \arrow[dl, "C_F"] \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{tikzcd}$$

Bridging morphisms and comorphisms of sites

We shall call a functor which both a morphism and a comorphism of sites a bimorphism of sites.

We have actually shown that the relationship between a morphism F (resp. comorphism G) of sites and the associated comorphism c_F (resp. morphism m_F) of sites is captured by the equivalence

$$\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \simeq \mathbf{Sh}((c_F \downarrow 1_{\mathcal{D}}), \overline{\tilde{K}})$$

(resp.

$$\mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \simeq \mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}))$$

of toposes over $\mathbf{Sh}(\mathcal{C}, J)$ induced by bimorphism of sites w_F (resp. z_G) over $\mathcal{C}$.

In fact, F and c_F (resp. G and m_G) are not adjoint to each other in a concrete sense (that is, at the level of sites); nonetheless, they become 'abstractly' adjoint in the world of toposes since toposes naturally attached to such categories are equivalent.

These correspondences actually yield a dual adjunction between a category of morphisms of sites from a given site and a category of comorphisms of sites towards that site.

Continuous functors

Another important class of functors between sites is that of continuous ones:

Definition (Grothendieck)

Given sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be (J, K)-continuous, or simply, continuous, if the functor

$$D_A := (- \circ A^{\text{op}}) : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

The property of continuity of a functor can be interpreted as a form of cofinality; in fact, we have shown that it can be explicitly characterized in terms of "relative cofinality conditions" (introduced in the paper *Denseness conditions, morphisms and equivalences of toposes*).

Classifying essential morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be essential if its inverse image f^* has a left adjoint, denoted by $f_!$ and called its essential image.

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos. Let $\mathbf{Geom}_{\text{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E})$ be the category of essential geometric morphisms, and $\mathbf{Com}_{\text{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\text{can}}_{\mathcal{E}}))$ the category of J-continuous comorphisms of sites $(\mathcal{C}, J) \to (\mathcal{E}, J^{\text{can}}_{\mathcal{E}})$. Then we have an equivalence

$$\mathbf{Geom}_{\text{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E}) \simeq \mathbf{Com}_{\text{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\text{can}}_{\mathcal{E}}))$$

sending an essential geometric morphism $f = (f_! \dashv f^* \dashv f_*)$ to the comorphism of sites $f_! \circ l$ and a J-continuous comorphism of sites A to the geometric morphism C_A induced by it.

Remark

We have shown that if π is a fibration then C_π is not only essential, but even locally connected.

Fibrations as comorphisms of sites

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K in $\mathcal{D}$, there is a smallest Grothendieck topology M^A_K on $\mathcal{C}$ which makes A a comorphism of sites to $(\mathcal{D}, K)$.

Proposition

If A is a fibration, the topology M^A_K admits the following simple description: a sieve R is M^A_K-covering if and only if the collection of cartesian arrows in R is sent by A to a K-covering family.

We shall call M^A_K the Giraud topology induced by K, in honour of Jean Giraud, who used it for constructing the classifying topos $\mathbf{Sh}(\mathcal{C}, M^A_K)$ of a stack A on $(\mathcal{D}, K)$.

Proposition

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a continuous comorphism of sites $(\mathcal{C}, M^A_K) \to (\mathcal{C}', M^{A'}_K)$.

In particular, a fibration $A : \mathcal{C} \to \mathcal{D}$ yields a continuous comorphism of sites $(\mathcal{C}, M^A_K) \to (\mathcal{D}, K)$ for any Grothendieck topology K on $\mathcal{D}$.

Giraud topologies

The study of the Giraud topology can provide insights on the given fibration. As a basic example of this, under the assumption that J is subcanonical, the property of being a prestack can be checked directly by analysing the Giraud topology:

Proposition (O.C. and R.Z.)

Consider a subcanonical site $(\mathcal{C}, J)$ and a cloven fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a prestack if and only if the Giraud topology M_J^p is subcanonical.

We actually have a Giraud topology functor

$$\mathfrak{G} : \mathbf{Cat}/\mathcal{C} \to \mathbf{Com}/(\mathcal{C}, J),$$

mapping $[p : \mathcal{E} \to \mathcal{C}]$ to $p : (\mathcal{E}, M_J^p) \to (\mathcal{C}, J)$.

By the above results, this functor actually takes values in the subcategory of continuous comorphisms of sites.

Indexed categories and fibrations

The language in which we shall work for developing relative topos theory is that of indexed categories and fibrations.

- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Ind}_{\mathcal{C}}$ the 2-category of $\mathcal{C}$-indexed categories: it is the 2-category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]_{ps}$ whose 0-cells are the pseudofunctors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, whose 1-cells are the pseudonatural transformations and whose 2-cells are the modifications between them.
- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Fib}_{\mathcal{C}}$ the 2-category of fibrations over $\mathcal{C}$: it is the sub-2-category of $\mathbf{CAT}/\mathcal{C}$ whose 0-cells are the (Street) fibrations $p : \mathcal{D} \to \mathcal{C}$, whose 1-cells are the morphisms of fibrations (with a 'commuting' isomorphism) and whose 2-cells are the natural transformations between them.

  We shall denote by $\mathbf{cFib}_{\mathcal{C}}$ the full sub-2-category of cloven fibrations (i.e. fibrations equipped with a cleavage).

It is well-known that indexed categories and fibrations are in equivalence with each other:

Theorem

For any category $\mathcal{C}$, there is an equivalence of 2-categories between $\mathbf{Ind}_{\mathcal{C}}$ and $\mathbf{cFib}_{\mathcal{C}}$, one half of which is given by the Grothendieck construction and whose other half is given by the functor taking the fibers at the objects of $\mathcal{C}$.

The notion of stack

Definition

Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a J-prestack (resp. J-stack) if for every J-sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$- \circ \textstyle\int m_S : \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\textstyle\int S, \mathcal{D})$$

is full and faithful (resp. an equivalence).

Stacks over a site $(\mathcal{C}, J)$ form a 2-full and faithful subcategory of $\mathbf{Ind}_{\mathcal{C}}$, which we will denote by $\mathbf{St}(\mathcal{C}, J)$.

The notion of stack on a site is a higher-categorical generalization of that of sheaf on that site:

Proposition

Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$: then P is J-separated (resp. J-sheaf) if and only if the fibration $\int P \to \mathcal{C}$ is a J-prestack (resp. J-stack).

We can rewrite the condition for a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ to be a J-prestack (resp. J-stack) in the language of indexed categories, as the requirement that for every sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(y_{\mathcal{C}}(X), \mathbb{D}) \xrightarrow{- \circ m_S} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{D})$$

be full and faithful (resp. an equivalence), where both $y_{\mathcal{C}}(X)$ and S are interpreted as discrete $\mathcal{C}$-indexed categories.

Stacks for relative topos theory

The role of stacks in our approach to relative topos theory is two-fold:

- On the one hand, the notion of stack represents a higher-order categorical generalization of the notion of sheaf. Accordingly, categories of stacks on a site represent higher-categorical analogues of Grothendieck toposes. One can thus expect to be able to lift a number of notions and constructions pertaining to sheaves (resp. Grothendieck toposes) to stacks (resp. categories of stacks on a site).
- On the other hand, stacks on a site $(\mathcal{C}, J)$ generalize internal categories in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Since (ordinary) categories can be endowed with Grothendieck topologies, so stacks on a site can also be endowed with suitable analogues of Grothendieck topologies. This leads to the notion of *site relative to a base topos*, which is crucial for developing relative topos theory.

Remark

Every stack is equivalent to a split stack, and hence to an internal category, but most stacks naturally arising in the mathematical practice are not split (think, for instance, of the canonical stack of a topos).

The big picture

Our theory is based on a network of 2-adjunctions (for any small site $(\mathcal{C}, J)$):

$$
\begin{tikzcd}
\mathbf{Ind}_{\mathcal{C}} \arrow[rr, shift left=2, "\Lambda"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left=2, "s_J"] \arrow[d, phantom, "\vdash"] & & \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[ll, shift left=2, "\Gamma"] \\
\mathbf{St}(\mathcal{C}, J) \arrow[u, hook, shift left=2] \arrow[rr, shift left=2, "\Lambda'"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left=2, "E \circ \Lambda'"] \arrow[d, phantom, "\vdash"] & & \mathbf{EssTopos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \arrow[u, hook] \arrow[ll, shift left=2, "\Gamma'"] \arrow[dll, shift right=2, "E"'] \arrow[dll, phantom, "\dashv"] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[u, hook, shift left=2] \arrow[urr, shift right=2, "L"'] & &
\end{tikzcd}
$$

In this diagram s_J denotes the stackification functor, **Topos** the category of Grothendieck toposes and geometric morphisms and **EssTopos** the full subcategory on the essential geometric morphisms.

- The functor E sends an essential geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the object $f_!(1_{\mathscr{E}})$ (where $f_!$ is the left adjoint to the inverse image f^* of f).
- The functor L sends an object P of $\mathbf{Sh}(\mathcal{C}, J)$ to the canonical local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathbf{Sh}(\mathcal{C}, J)$.

Pseudo-Kan extensions

Proposition

Denote by $\textbf{Ind}^s_{\mathcal{C}}$ *the sub-2-category of* $\textbf{Ind}_{\mathcal{C}}$ *of pseudofunctors with values in* **Cat** *(i.e. 'small' $\mathcal{C}$-indexed categories). Consider any functor* $F : \mathcal{C} \to \mathcal{D}$ *and the direct image 2-functor*

$$F^* : \textbf{Ind}^s_{\mathcal{D}} \to \textbf{Ind}^s_{\mathcal{C}}$$

which acts by precomposition with F^{op}*. The 2-functor* F^* *has both a left and a right 2-adjoint, denoted respectively by* $\mathrm{Lan}_{F^{\mathrm{op}}}$ *and* $\mathrm{Ran}_{F^{\mathrm{op}}}$*, which act as follows:*

- *for any* D *in* $\mathcal{D}$ *denote by* $\pi^D_F : (D{\downarrow}F) \to \mathcal{C}$ *the canonical projection functor: then for* $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to \textbf{Cat}$*, its image* $\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to \textbf{Cat}$ *is defined componentwise as*

$$\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \mathrm{colim}_{ps}\left((D{\downarrow}F)^{\mathrm{op}} \xrightarrow{(\pi^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \textbf{Cat}\right)$$

- *for any* D *in* $\mathcal{D}$ *denote by* $\pi'^D_F : (F{\downarrow}D) \to \mathcal{C}$ *the canonical projection functor: then for* $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to \textbf{Cat}$*, its image* $\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to \textbf{Cat}$ *is defined componentwise as*

$$\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \lim_{ps}\left((F{\downarrow}D)^{\mathrm{op}} \xrightarrow{(\pi'^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \textbf{Cat}\right)$$

Direct and inverse images of stacks

Proposition (O.C. and R.Z.)

Consider two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

- *Then F is (J, K)-continuous functor if and only if $F^* : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}}$ restricts to a 2-functor $\mathbf{St}(\mathcal{D}, K) \to \mathbf{St}(\mathcal{C}, J)$.*
- *If F is a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$, or more generally a (J, K)-continuous functor, it induces a 2-adjunction*

$$\begin{tikzcd} \mathbf{St}^s(\mathcal{C}, J) \arrow[r, bend left, "\mathbf{St}(F)^*"] \arrow[r, phantom, "\perp"] & \mathbf{St}^s(\mathcal{D}, K) \arrow[l, bend left, "\mathbf{St}(F)_*"] \end{tikzcd} ,$$

whose pair we shall refer to simply by $\mathbf{St}(F)$.
- *The 2-functor $\mathbf{St}(F)_*$ is called the direct image of stacks along F and acts as the precomposition*

$$F^* := (- \circ F^{\mathrm{op}}) : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}};$$

In terms of fibrations, a stack $q : \mathcal{E} \to \mathcal{D}$ is mapped by $\mathbf{St}(F)_*$ to its strict pseudopullback $p : \mathcal{P} \to \mathcal{C}$ along F.

Direct and inverse images of stacks

- The left adjoint $\mathbf{St}(F)^*$ is the inverse image of stacks along F and acts as the composite

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K),$$

where s_K denotes the stackification functor. In terms of fibrations, a stack $p : \mathcal{P} \to \mathcal{C}$ is mapped by $\mathbf{St}(F)^*$ to the stackification of its inverse image $\mathrm{Lan}_{F^{\mathrm{op}}}([p])$ along F, which can be computed as a localization as follows. Consider the fibration of generalized elements

$$(1_{\mathcal{D}} {\downarrow} (F \circ p)) \xrightarrow{r} \mathcal{D}$$

of the functor $F \circ p$, whose objects are arrows $[d : D \to (F \circ p)(U)]$ of $\mathcal{D}$, and whose morphisms

$$(e, \alpha) : [d' : D' \to (F \circ p)(V)] \to [d : D \to (F \circ p)(U)]$$

are indexed by an arrow $e : D' \to D$ in $\mathcal{D}$ and an arrow $\alpha : V \to U$ in $\mathcal{P}$ such that $(F \circ p)(\alpha) \circ d' = d \circ e$. Consider the class of arrows

$$S := \{(e, \alpha) : [d'] \to [d] \mid (e, \alpha) \text{ } r\text{-vertical}, \alpha \text{ cartesian in } \mathcal{P}\} :$$

then

$$\mathrm{Lan}_{F^{\mathrm{op}}}([p]) \simeq (1_{\mathcal{D}} {\downarrow} (F \circ p))[S^{-1}] .$$

Direct and inverse images of stacks

In a similar way to morphisms of sites, comorphisms of sites also induce an adjunction between categories of stacks:

Proposition (O.C. and R.Z.)

Consider a comorphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$*: it induces a 2-adjunction*

$$\begin{tikzcd} \mathbf{St}^s(\mathcal{D}, K) \arrow[r, bend left, "(C_F^{\mathbf{St}})^*"] \arrow[r, phantom, "\perp"] & \mathbf{St}^s(\mathcal{C}, J) \arrow[l, bend left, "(C_F^{\mathbf{St}})_*"] \end{tikzcd} ,$$

whose pair we shall refer to by $C_F^{\mathbf{St}}$.

- *The right adjoint* $(C_F^{\mathbf{St}})_*$ *acts by restriction of the right pseudo-Kan extension* $\mathrm{Ran}_{F^{\mathrm{op}}}$ *to stacks;*
- *The left adjoint* $(C_F^{\mathbf{St}})^*$ *acts as the composite 2-functor*

  $$\mathbf{St}^s(\mathcal{D}, K) \xrightarrow{i_K} \mathbf{Ind}_{\mathcal{D}}^s \xrightarrow{F^*} \mathbf{Ind}_{\mathcal{C}}^s \xrightarrow{s_J} \mathbf{St}^s(\mathcal{C}, J),$$

  where $F^* := (- \circ F^{\mathrm{op}})$.
- *If* F *is also* *continuous* *the* $C_F^{\mathbf{St}}$ *also has a left adjoint* $(C_F^{\mathbf{St}})_!$ *given by the composite 2-functor*

  $$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}_{\mathcal{C}}^s \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}_{\mathcal{D}}^s \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K) .$$

Relative 'presheaf toposes'

Given a $\mathcal{C}$-indexed category $\mathbb{D}$, we denote by $\mathcal{G}(\mathbb{D})$ the fibration on $\mathcal{C}$ associated with it (through the Grothendieck construction) and by $p_{\mathbb{D}}$ the canonical projection functor $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

Proposition (O.C. and R.Z.)

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}$ a $\mathcal{C}$-indexed category and $\mathbb{D}^V$ be the opposite indexed category of $\mathbb{D}$ (defined by setting, for each $c \in \mathcal{C}$, $\mathbb{D}^V(c) = \mathbb{D}(c)^{op}$). Then we have a natural equivalence

$$\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}) .$$

This proposition shows that the topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of $\mathbb{D}$, which we call the Giraud topos of $\mathbb{D}$, can indeed be seen as the "topos of relative presheaves on $\mathbb{D}$".

Giraud toposes as weighted colimits

We have shown that, for any $\mathbb{D}$, the Giraud topos $C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \to \mathbf{Sh}(\mathcal{C}, J)$ can be naturally seen as a weighted colimit of a diagram of étale toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

$$
\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}/X, J_X) \arrow[d, bend right=40, "\lambda_{(X,V)}"'] \arrow[d, bend left=40, "\lambda_{(X,U)}"] \arrow[d, phantom, "\xLeftarrow{\lambda_{(X,a)}}"] \arrow[dr, phantom, "\cong"] & \mathbf{Sh}(\mathcal{C}/Y, J_X) \arrow[l, "C_{\Sigma_y}"'] \arrow[dl, bend left, "\lambda_{(Y,(\mathbb{D}(y)(U)))}"] \\
\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) &
\end{tikzcd}
$$

where $y : Y \to X$ and $a : U \to V$ are arrows respectively in $\mathcal{C}$ and in $\mathbb{D}(X)$, the legs $\lambda_{(X,U)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of the cocone are the morphisms $C_{\xi_{(X,U)}}$ induced by the morphisms of fibrations $\xi_{(X,U)} : \mathcal{C}/X \to \mathcal{D}$ over $\mathcal{C}$ given by the fibered Yoneda lemma, and the functor $\Sigma_y : \mathcal{C}/Y \to \mathcal{C}/X$ are given by composition with y.

The fundamental adjunction

A fundamental result in the theory of sheaves is the classical adjunction

$$\begin{tikzcd} \mathbf{Psh}(X) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathbf{Top}/X \arrow[l, bend left, "\Gamma"] \end{tikzcd} .$$

between presheaves on a topological space X and bundles over it (i.e. continuous maps with codomain X).

How to possibly generalize this adjunction to the setting of arbitrary sites has been an open problem for many years.

With Riccardo Zanfa, we have managed to establish such a generalization, not only for presheaves, but for arbitrary indexed categories. This result has several applications to the theory of sheaves and stacks.

The fundamental adjunction

For any small-generated site $(\mathcal{C}, J)$, there is a 2-adjunction between cloven fibrations over $\mathcal{C}$ and toposes over $\mathbf{Sh}(\mathcal{C}, J)$ (which in fact precisely expresses the universal property of the above weighted colimit):

Theorem (O.C and R.Z.)

For any small-generated site $(\mathcal{C}, J)$, the two pseudofunctors

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}_{\mathcal{C}} \to \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\Big[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\phi)} [q : \mathcal{E} \to \mathcal{C}]\Big] \mapsto \Big[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\phi)} [\mathrm{Gir}_J(q)]\Big],$$

and

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}_{\mathcal{C}} \simeq \mathbf{cFib}_{\mathcal{C}},$$

$$[E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)] \mapsto \big[\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{\mathrm{op}} \to \mathbf{CAT}\big]$$

are the two components of a 2-adjunction

$$\begin{tikzcd} \mathbf{cFib}_{\mathcal{C}} \arrow[r, bend left, "\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \arrow[r, phantom, "\perp"] & \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \arrow[l, bend left, "\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}"] \end{tikzcd}$$

Remark

Since $\mathrm{Gir}_J(p) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathcal{D}^V, \mathcal{S}_{(\mathcal{C},J)})$, the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ has a similar behavior to that of a dualizing object for the adjunction $\Lambda \dashv \Gamma$.

The discrete setting

Proposition (O.C. and R.Z.)

Consider a small-generated site $(\mathcal{C}, J)$:

- *There is an adjunction of 1-categories*

$$\begin{tikzcd}[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \arrow[r, bend left, "\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}"] \arrow[r, phantom, "\perp"] & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \arrow[l, bend left, "\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}"]\end{tikzcd} .$$

- *The functor $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ maps a presheaf P to $\prod_{a_J(P)} : \mathbf{Sh}(\mathcal{C}, J)/a_J(P) \to \mathbf{Sh}(\mathcal{C}, J)$ or, in terms of comorphisms of sites, to $\Lambda(P) := [C_{p_P} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)]$ and $\Lambda(g) := C_{\int g} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$.*
- *The functor $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ acts like a Hom-functor by mapping an object $[F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)]$ of $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ to the presheaf*

$$\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{F}) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} .$$

The general presheaf-étale adjunction

- The image of $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ factors through $\mathbf{Topos}^{\textit{étale}}/\mathbf{Sh}(\mathcal{C},J)$, and the image of $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ factors through $\mathbf{Sh}(\mathcal{C},J)$;
- The fixed points of $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)$ are precisely the étale geometric morphisms, while those of $[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]$ are the J-sheaves.
- The adjunction $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ restricts to an equivalence

$$\mathbf{Sh}(\mathcal{C},J) \simeq \mathbf{Topos}^{\textit{étale}}/_1\mathbf{Sh}(\mathcal{C},J)\ .$$

- The composite functor $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ is naturally isomorphic to the sheafification functor

$$i_J a_J : [\mathcal{C}^{\mathrm{op}},\mathbf{Set}] \to \mathbf{Sh}(\mathcal{C},J) \to [\mathcal{C}^{\mathrm{op}},\mathbf{Set}];$$

Some applications

The presheaf-bundle adjunction for topological spaces is useful mostly becase it provides a geometric interpretation of several fundamental constructions on (pre)sheaves, such as direct and inverse images, as well as the sheafification process, in the language of fibrations.

Thanks to our site-theoretic generalization, we can extend these techniques to arbitrary presheaves. In particular, we obtain the following results:

- For any $c \in \mathcal{C}$, the elements $a_J(P)(c)$ of the J-sheafification of a given presheaf P can be identified with the geometric morphisms over $\mathbf{Sh}(\mathcal{C}, J)$ from $\mathbf{Sh}(\mathcal{C}/c, J_c)$ to $\mathbf{Sh}(\int P, J_P)$, all of which can be locally represented as being induced by morphisms of fibrations.

  This is strictly related to the construction of $a_J(P)(c)$ in terms of locally matching families of elements of P.

Direct and inverse images of sheaves in terms of fibrations

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ and two presheaves $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and $Q : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ with associated fibrations $\pi_P : \int P \to \mathcal{C}$ and $\pi_Q : \int Q \to \mathcal{D}$,
  - the fibration corresponding to the direct image presheaf $Q \circ F^{\mathrm{op}}$ is computed as the strict pullback of π_Q along F:

$$\begin{tikzcd} \int(F^*(Q)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int Q \arrow[d, "\pi_Q"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

  - If F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then, for any J-sheaf P on $\mathcal{C}$, the inverse image $\mathbf{Sh}(F)^*(P)$ coincides with the discrete part of the K-comprehensive factorization of the composite functor $F \circ \pi_P$.

We have also established natural analogues of these results in the context of stacks.

Applications to pullbacks of toposes

Given a continuous function $f : X \to Y$ between topological spaces, the inverse image $\mathbf{Sh}(f)^*$ of the geometric morphism

$$\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$$

induced by f corresponds under the equivalences

$$\mathbf{Sh}(X) \simeq \mathbf{Etale}/X$$

and

$$\mathbf{Sh}(Y) \simeq \mathbf{Etale}/Y$$

to the pullback operation along f in the category **Top** of topological spaces.

The following result represents the topos-theoretic analogue of this result (note how geometric morphisms play the role of continuous maps):

Proposition (O.C. and R.Z.)

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites and $\mathbb{D}$ a small J-stack: then the square

$$\begin{tikzcd} \mathrm{Gir}_K(\mathbf{St}(F)^*(\mathbb{D})) \arrow[r] \arrow[d, "C_{p_{(\mathbf{St}(F)^*(\mathbb{D}))}}"'] & \mathrm{Gir}_J(\mathbb{D}) \arrow[d, "C_{p_{\mathbb{D}}}"] \\ \mathbf{Sh}(\mathcal{D}, K) \arrow[r, "\mathbf{Sh}(F)"'] & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

is a pullback of toposes.

Applications to pullbacks of toposes

As shown by the following proposition, in the case of geometric morphism induced by a continuous comorphism of sites, it is possible to compute the following pullback of toposes very simply already at the site level:

Proposition (O.C. and R.Z.)

If F is a continuous comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then, for any small K-stack $\mathbb{D}$ on $\mathcal{D}$, the diagram

$$\begin{tikzcd}
\mathcal{G}((C_F^{\mathbf{St}})^*(\mathbb{D})) \arrow[r] \arrow[d, "p_{(C_F^{\mathbf{St}})^*(\mathbb{D})}"'] & \mathcal{G}(\mathbb{D}) \arrow[d, "p_{\mathbb{D}}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

is a pseudopullback in **Cat**, *which is sent by the 2-functor C to a pullback in* **Topos**:

$$\begin{tikzcd}
\mathrm{Gir}_J((C_F^{\mathbf{St}})^*(\mathbb{D})) \arrow[r] \arrow[d, "C_{p_{(C_F^{\mathbf{St}})^*(\mathbb{D})}}"'] & \mathrm{Gir}_K(\mathbb{D}) \arrow[d, "C_{p_{\mathbb{D}}}"] \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[r, "C_F"'] & \mathbf{Sh}(\mathcal{D}, K)
\end{tikzcd}$$

Relative sheaf toposes

As any Grothendieck topos is a subtopos of a presheaf topos, so any relative topos should be a subtopos of a relative presheaf topos. This motivates the following

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A site relative to $(\mathcal{C}, J)$ is a pair consisting of a $\mathcal{C}$-indexed category $\mathbb{D}$ and a Grothendieck topology K on $\mathcal{G}(\mathbb{D})$ which contains the Giraud topology $M_J^{p_{\mathbb{D}}}$.

The topos of sheaves on such a relative site $(\mathbb{D}, K)$ is defined to be the geometric morphism

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by the comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Remark

Not every Grothendieck topology on K can be generated starting by horizontal or vertical data (that is, by sieves generated by cartesian arrows or entirely lying in some fiber), but many important relative topologies naturally arising in practice are of this form.

Examples of relative topologies

- The relative topology on the canonical stack of a geometric morphism (which allows one to represent *any* relative topos as the topos of sheaves on a relative site).
- The Giraud topology is an example of a relative topology generated by horizontal data.
- The total topology of a fibered site, in the sense of Grothendieck, is generated by vertical data.

We have shown that, for a wide class of relative topologies generated by horizontal and vertical data, one can describe bases for them consisting of multicompositions of horizontal families with vertical families.

From the ordinary to the relative setting

ABSOLUTE	RELATIVE
Topos $\mathcal{E}$	Relative topos : $f : \mathcal{F} \to \mathcal{E}$
Geometric morphism $g : \mathcal{E} \to \mathcal{E}'$	Relative geometric morphism $g : [f] \to [f']$
Categories and functors	Fibrations and their morphisms
Site $(\mathcal{D}, K)$	Relative site $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$
Morphism of sites	Morphism of relative sites
Presheaf toposes	Giraud toposes
Sheaf toposes	Toposes of sheaves on a relative site
Canonical site $(\mathcal{E}, J^{\mathcal{E}}_{can})$	Canonical relative site $((1 \downarrow f^*), J_f)$
Canonical functor $l : (\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{D}, K)$	Canonical morphism $\eta_D : (\mathcal{D}, K) \to (\mathbf{Sh}(\mathcal{D}, K) \downarrow C_p^*)$

Relative Diaconescu's equivalence

A fundamental result in topos theory is Diaconescu's theorem, providing an equivalence between geometric morphisms to a sheaf topos $\mathbf{Sh}(\mathcal{D}, K)$ and K-continuous flat functors on $\mathcal{D}$.

The following result represents its generalisation in the setting of relative toposes:

Theorem (L.B. and O.C.)

Let $f : \mathcal{F} \to \mathbf{Sh}(\mathcal{C}, J)$ be a $\mathbf{Sh}(\mathcal{C}, J)$-topos and $\mathbb{D}$ a $\mathcal{C}$-indexed category. Then we have an equivalence

$$\mathbf{Geom}_{\mathbf{Sh}(\mathcal{C},J)}([f], [C_{p_{\mathbb{D}}}]) \simeq \mathbf{Flat}_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{G}(\mathbb{D}), (1_{\mathcal{F}} \downarrow f^* l_J))$$

between the category of geometric morphisms, relative to $\mathbf{Sh}(\mathcal{C}, J)$, from f to $C_{p_{\mathbb{D}}}$, and the category of flat relative to $\mathbf{Sh}(\mathcal{C}, J)$ functors from $\mathcal{G}(\mathbb{D})$ to $(1_{\mathcal{F}} \downarrow f^* l_J)$.
If K is a Grothendieck topology on $\mathcal{G}(\mathbb{D})$ containing the Giraud topology $J_{\mathbb{D}}$ then the above equivalence restricts to an equivalence

$$\mathbf{Geom}_{\mathbf{Sh}(\mathcal{C},J)}([f], [C_{p^K_{\mathbb{D}}}]) \simeq \mathbf{Flat}^K_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{G}(\mathbb{D}), (1_{\mathcal{F}} \downarrow f^* l_J)),$$

where $\mathbf{Flat}^K_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{G}(\mathbb{D}), (1_{\mathcal{F}} \downarrow f^* l_J))$ is the full subcategory of the category $\mathbf{Flat}_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{G}(\mathbb{D}), (1_{\mathcal{F}} \downarrow f^* l_J))$ on the K-continuous functors.

Morphisms of relative sites

As in the ordinary setting, (relative) flat functors can be characterized in terms of (relative) morphisms of sites.

Definition (L.B. and O.C.)

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}, \mathbb{D}$ two $\mathcal{C}$-indexed categories and K, K' Grothendieck topologies respectively on $\mathcal{G}(\mathbb{D})$ and $\mathcal{G}(\mathbb{D}')$ which contain the Giraud topologies induced by J. A morphism of relative sites $(\mathcal{G}(\mathbb{D}), K) \to (\mathcal{G}(\mathbb{D}'), K')$ over $(\mathcal{C}, J)$ is a morphism of fibrations over $\mathcal{C}$ which is moreover a morphism of ordinary sites $(\mathcal{G}(\mathbb{D}), K) \to (\mathcal{G}(\mathbb{D}'), K')$.

Proposition (L.B. and O.C.)

Let $f : \mathcal{F} \to \mathbf{Sh}(\mathcal{C}, J)$ be a $\mathbf{Sh}(\mathcal{C}, J)$-topos and $\mathbb{D}$ a $\mathcal{C}$-indexed category. Then a functor $\mathcal{G}(\mathbb{D}) \to (1_{\mathcal{F}} \downarrow f^* l_J)$ is flat and K-continuous relative to the topos $\mathbf{Sh}(\mathcal{C}, J)$ if and only if it yields a morphism of relative sites $(\mathcal{G}(\mathbb{D}), K) \to ((1_{\mathcal{F}} \downarrow f^* l_J), J_f)$.

Fibred sites

The following slides will be devoted to presenting a way for representing relative toposes which naturally generalizes the construction of the topos of sheaves on a locale, and which is particularly effective for describing in a simple way the morphisms between relative toposes.

Recall that, given locales L and L', the morphisms $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ correspond exactly to the locale homomorphisms $L \to L'$.

Our representation will be based on the concept of existential fibred site.

By using this notion, we shall be able to describe the morphisms between two relative toposes as morphisms between the associated existential fibred sites.

Two corollaries of relative Diaconescu

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ and $f' : \mathcal{F}' \to \mathcal{E}$ be geometric morphisms towards the same base topos $\mathcal{E}$. Then we have an equivalence of categories

$$\mathbf{Geom}_{\mathcal{E}}([f],[f']) \simeq \mathbf{Fib}^{\text{cart,cov}}_{\mathcal{E}}(((1_{\mathcal{F}'} \downarrow f'^{*}), J_{f'}), ((1_{\mathcal{F}} \downarrow f^{*}), J_{f})),$$

where $\mathbf{Fib}^{\text{cart,cov}}_{\mathcal{E}}(((1_{\mathcal{F}'} \downarrow f'^{*}), J_{f'}), ((1_{\mathcal{F}} \downarrow f^{*}), J_{f}))$ is the category of morphisms of fibrations over $\mathcal{E}$ which are cartesian at each fiber and cover-preserving.

Corollary

Let $\mathcal{E}$ be a Grothendieck topos and L, L' internal locales in $\mathcal{E}$. Then we have an equivalence of categories

$$\mathbf{Geom}_{\mathcal{E}}(\mathbf{Sh}_{\mathcal{E}}(L), \mathbf{Sh}_{\mathcal{E}}(L')) \simeq \mathbf{Loc}_{\mathcal{E}}(L, L'),$$

where $\mathbf{Loc}_{\mathcal{E}}(L, L')$ is the category of morphisms of internal locales from L to L' in $\mathcal{E}$.

Fibred sites

Definition

Let $(\mathcal{C}, J)$ be a small-generated site.

(a) A fibred site over $\mathcal{C}$ is an indexed category $L : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ taking values in the category of small-generated sites and morphisms of sites between them; we shall denote by J^L_e the Grothendieck topology on the fiber $L(e)$.

(b) A fibred site over $(\mathcal{C}, J)$ is a fibred site over $\mathcal{C}$ which is J-reflecting in the sense that for any J-covering family S on an object c of $\mathcal{C}$ and any family T of arrows with common codomain in the category $L(c)$, if $L(f)(T)$ is $J^L_{\mathrm{dom}(f)}$-covering in the category $L(\mathrm{dom}(f))$ for every $f \in S$ then T is J^L_c-covering.

(c) A J-reflecting fibred site $L : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ over $(\mathcal{C}, J)$ is said to be existential if for any arrow $a : E' \to E$ in $\mathcal{C}$, the transition functor $L(a) : L(E) \to L(E')$ has a cover-preserving left adjoint, denoted $\exists_a : L(E') \to L(E)$ (which is therefore a comorphism of sites $(L(E'), J^L_{E'}) \to (L(E), J^L_E)$), and the following two conditions (where, for any f, η_f denotes the unit of the adjunction $\exists_f \dashv L(f)$) are satisfied:

Existential fibred sites

(i) Relative Beck-Chevalley condition:
For any arrows $c : V \to Z$ and $d : W \to Z$ in $\mathcal{C}$ with common codomain and any $l \in L(V)$, the family of arrows

$$\{\overline{\widetilde{L(a)(\eta_c(l))}} : (\exists_b)(L(a)(l)) \to L(d)(\exists_c(l)) \mid (a, b) \in B_{(c,d)}\}$$

is J^L_W-covering, where $B_{(c,d)}$ is the collection of spans $(a : U \to V, b : U \to W)$ such that $c \circ a = d \circ b$

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "b"'] & V \arrow[d, "c"] \\ W \arrow[r, "d"] & Z \end{tikzcd}$$

and $\overline{\widetilde{L(a)(\eta_c(l))}}$ is the transpose of the arrow

$$\widetilde{L(a)(\eta_c(l))} : L(a)(l) \to L(b)(L(d)(\exists_c(l)))$$

given by the composite of the arrow $L(a)(\eta_c(l))$ with the inverse of the isomorphism $L(b)(L(d)(\exists_c(l))) \to L(a)(L(c)(\exists_c(l)))$ resulting from the equality $c \circ a = d \circ b$ in light of the pseudofonctoriality of L.

Existential fibred sites

(ii) Relative Frobenius condition: For any arrows $f : E \to E'$ in $\mathcal{C}$, any $l \in L(E')$ and any arrow $\alpha : l' \to \exists_f(l)$, the family of arrows $\{\overline{\delta} : \exists_f(m) \to l' \mid (\delta, \rho) \in Q_{(f,\alpha)}\}$ is $J^L_{E'}$-covering, where $Q_{(f,\alpha)}$ is the collection of span of arrows $(\rho : m \to l, \delta : m \to L(f)(l'))$ in $L(E)$ which make the rectangle

$$\begin{tikzcd} m \arrow[r, "\rho"] \arrow[d, "\delta"'] & l \arrow[d, "\eta_f(l)"] \\ L(f)(l') \arrow[r, "L(f)(\alpha)"'] & L(f)(\exists_f(l)) \end{tikzcd}$$

commute.

Remark

One can generalize the notion of fibred site by simply requiring the transition morphisms to be cover-preserving (rather than morphisms of sites). The theorem below about the existential topology (see below) remains valid, but the results below on fibers of existential toposes require the stronger assumptions.

The fibred site of a geometric morphism

Definition

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. The existential fibred site of f is the indexed functor $L_f : \mathcal{E}^{\text{op}} \to \text{Cat}$ sending any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ endowed with its canonical topology (for any arrow $k : E' \to E$ in $\mathcal{E}$, the pullback functor

$$L_f(k) := (f^*(k))^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$$

has a left adjoint

$$\exists_k : \mathcal{F}/f^*(E') \to \mathcal{F}/f^*(E)$$

given by composition with $f^*(k)$.

If $(\mathcal{C}, J)$ is a site of definition for $\mathcal{E}$, the composite of L_f with the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ is also called the existential fibred site of f.

Existential toposes

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site and $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ a J-reflecting fibred site over $\mathcal{C}$. Then L is existential if and only if the families on the category $\mathcal{G}(L)$ of the form

$$\{(e_i, \alpha_i) : (E_i, l_i) \to (E, l) \mid i \in I\}$$

where the family $\{\overline{\alpha_i} : \exists_{e_i}(l_i) \to l \mid i \in I\}$ is J^L_E-covering are the covering families for a Grothendieck topology J^{ext}_L, called the existential topology, on $\mathcal{G}(L)$.

Moreover, if L is an existential fibred site over $(\mathcal{C}, J)$, the existential topology J^{ext}_L contains the Giraud topology induced by J.

The relative topos

$$C_{p_L} : \mathbf{Sh}(\mathcal{G}(L), J^{\text{ext}}_L) \to \mathbf{Sh}(\mathcal{C}, J)$$

is called the existential topos of L.

Existential toposes

Proposition

- *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then, under the identification*

  $$(1 \downarrow f^*) \cong \mathcal{G}(L_f),$$

  the topology J_f on $(1 \downarrow f^*)$, that is, the relative topology of f, corresponds to the existential topology $J^{ext}_{L_f}$ on $\mathcal{G}(L_f)$, where L_f is the existential fibred site of f.

- *Every internal locale L in a topos $\mathcal{E}$ yields an existential fibred preorder site over the canonical site of $\mathcal{E}$.*

  Moreover, for any $E \in \mathcal{E}$, the topos of canonical sheaves on the locale $L(E)$ can be recovered as the localic reflection of the slice at E of the existential topos associated with L.

Morphisms of existential fibred sites

Definition

Given a topos $\mathcal{E}$ and existential fibred sites L and L' over $\mathcal{E}$, a morphism $\alpha : L \to L'$ is a morphism of indexed categories which is cartesian and cover-preserving at each fiber and which commutes with the left adjoints $\exists_e$ for any arrow e in $\mathcal{E}$.

Theorem

Given relative toposes $[f : \mathcal{F} \to \mathcal{E}]$ and $[f' : \mathcal{F}' \to \mathcal{E}]$, the geometric morphisms $f \to f'$ over $\mathcal{E}$ correspond precisely to the morphisms of existential fibred sites $L_{f'} \to L_f$.

Remark

This is a natural generalization of the classical result stating that the geometric morphisms $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ correspond precisely to the frame homomorphisms $L' \to L$.

Fibers of existential toposes

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and L an existential fibred site over $(\mathcal{C}, J)$ and c an object of $\mathcal{C}$. Then the fibre $\mathbf{Sh}(\mathcal{G}(L), J_L^{\text{ext}})/C^*_{\pi_L}(l(c))$ *at c of the existential topos*

$$C_{\pi_L} : \mathbf{Sh}(\mathcal{G}(L), J_L^{\text{ext}}) \to \mathbf{Sh}(\mathcal{C}, J)$$

of L is equivalent to the topos of sheaves on the category $\mathcal{G}_c^{\text{ext}}(L)$ of elements of the functor $\mathrm{Hom}_{\mathcal{C}}(\pi_L(-), c)$, endowed with the Grothendieck topology $\widetilde{J}_c$ induced by J_L^{ext}.

For any arrow $k : c \to c'$ in $\mathcal{C}$, the pullback functor admits a left adjoint, given by the composition functor $\Sigma_{(C_{\pi_L})^*(l(k))}$ with $(C_{\pi_L})^*(l(k))$, which is induced by the comorphism of sites

$$E_k : \mathcal{G}_c^{\text{ext}}(L) \to \mathcal{G}_{c'}^{\text{ext}}(L)$$

given by composition with k.

Fibers of existential toposes

Proposition

For any object c of $\mathcal{C}$, the fiber at c of the existential topos of L is related to the topos of sheaves $\mathbf{Sh}(L(c), J_c^L)$ on the fiber of L at c via the hyperconnected (whence open) geometric morphism

$$\mathbf{Sh}(i_c) \cong C_{\mathrm{ext}_c} : \mathbf{Sh}(\mathcal{G}_c^{\mathrm{ext}}(L), \widetilde{J_c}) \to \mathbf{Sh}(L(c), J_c^L)$$

induced respectively by the morphism of sites

$$i_c : (L(c), J_c^L) \to (\mathcal{G}_c^{\mathrm{ext}}(L), \widetilde{J_c})$$

sending an object x of $L(c)$ to the object $((c, x), 1_c)$ of $\mathcal{G}_c^{\mathrm{ext}}(L)$, and by the (left adjoint) comorphism of sites

$$\mathrm{ext}_c : (\mathcal{G}_c^{\mathrm{ext}}(L), \widetilde{J_c}) \to (L(c), J_c^L)$$

sending an object $((d, y), f)$ of $\mathcal{G}_c^{\mathrm{ext}}(L)$ to the object $\exists_f(y)$ of $L(c)$. Moreover, for any arrow $k : c \to c'$ in $\mathcal{C}$, the following diagram of comorphism of sites commutes:

$$\begin{tikzcd}
(\mathcal{G}_c^{\mathrm{ext}}(L), \widetilde{J_c}) \arrow[r, "\mathrm{ext}_c"] \arrow[d, "E_k"] & (L(c), J_c^L) \arrow[d, "\exists_k"] \\
(\mathcal{G}_{c'}^{\mathrm{ext}}(L), \widetilde{J'_c}) \arrow[r, "\mathrm{ext}_{c'}"] & (L(c'), J_{c'}^L)
\end{tikzcd}$$

Characterization of internal locales

The following corollary gives a characterization of internal locales in a topos $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on an arbitrary, not necessarily cartesian, small-generated site $(\mathcal{C}, J)$:

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site. Then an internal locale in $\mathbf{Sh}(\mathcal{C}, J)$ is a functor $L : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ taking values in the subcategory of frames and frame homomorphisms which is a J-sheaf and, when considered as a fibred site (by endowing each frame with its canonical topology), is existential i.e. the following conditions are satisfied:

1. *Relative Beck-Chevalley condition: For any arrows $c : V \to Z$ and $d : W \to Z$ in $\mathcal{C}$ with common codomain and any $l \in L(V)$,*

$$L(d)(\exists_c(l)) = \bigvee_{(a,b)\in B_{(c,d)}} (\exists_b(L(a)(l))),$$

where $B_{(c,d)}$ is the collection of spans $(a : U \to V, b : U \to W)$ such that $c \circ a = d \circ c$;

2. *Frobenius reciprocity condition: for any $a : E \to E'$, $l \in L(E)$ and $l' \in L(E')$,*

$$\exists_a(L(a)(l') \wedge l) = \exists_a(l) \wedge l' .$$

Applications to logic

The idea of investigating logical theories by using a fibrational formalism dates back to Lawvere and his notion of (hyper)doctrine. More specifically:

- A first-order theory $\mathbb{T}$ over a signature Σ is represented as a fibred preorder $L_{\mathbb{T}}$ indexed by the category Sort_{Σ} of sorts of Σ, whose objects are the finite list of variables of sorts in Σ and whose arrows $\vec{x} \to \vec{y}$ are the maps from $\vec{y}$ to $\vec{x}$ which respect sorts.
- The Sort_{Σ}-indexed category $L_{\mathbb{T}}$ sends a context $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$ to the poset $L_{\mathbb{T}}(\vec{x})$ of $\mathbb{T}$-provable equivalence classes of first-order formulas over Σ in the context $\vec{x}$.
- The transition functors are given by substitution, and they have adjoints on both sides, given by existential quantification and universal quantification.

Alternative syntactic sites

From a topos-theoretic point of view, if $\mathbb{T}$ is a geometric theory then:

- the presheaf topos $[\mathrm{Sort}_\Sigma^{\mathrm{op}}, \mathbf{Set}]$ is the classifying topos $\mathcal{E}_{\mathbb{O}_\Sigma}$ of the empty theory $\mathbb{O}_\Sigma$ consisting of just the sorts of Σ;
- (The geometric version of) $L_\mathbb{T}$ is an internal locale in $[\mathrm{Sort}_\Sigma^{\mathrm{op}}, \mathbf{Set}]$;
- $\mathbb{T}$ is a localic expansion of $\mathbb{O}_\Sigma$, whence the canonical geometric morphism $\mathcal{E}_\mathbb{T} \to \mathcal{E}_{\mathbb{O}_\Sigma}$ between their classifying toposes is localic.
- Hence the classifying topos $\mathcal{E}_\mathbb{T}$ of $\mathbb{T}$ identifies with the existential topos associated with the fibred site $L_\mathbb{T}$; in particular, we obtain $(\mathcal{G}(L_\mathbb{T}), J^{\mathrm{ext}}_{L_\mathbb{T}})$ as an alternative syntactic site for the classifying topos of $\mathbb{T}$ (cf. J. Wrigley's talk at *Toposes Online* for more details about it).

As shown in the paper *Fibred sites and existential toposes*, many other alternative syntactic sites for the classifying topos of a theory can be obtained through the same method.

Completions of fibred preorder sites

It is possible to complete an arbitrary fibred preorder site to an internal locale:

Proposition

Let $(\mathbb{P}, K)$ be a fibered preordered site over a small-generated site $(\mathcal{C}, J)$. Then the canonical functor

$$\eta_{\mathbb{P}} : \mathbb{P} \to L_{C_{p_{\mathbb{P}}}},$$

where $L_{C_{p_{\mathbb{P}}}}$ is the internal locale associated with the geometric morphism $C_{p_{\mathbb{P}}}$, satisfies the universal property of the internal frame completion of $(\mathbb{P}, K)$.

It can be described as follows:

- *For any $c \in \mathcal{C}$, $L_{C_{p_{\mathbb{P}}}}(c)$ identifies with the frame*

$$\mathrm{ClSub}^{K}_{[\mathcal{G}(\mathbb{P})^{\mathrm{op}},\mathbf{Set}]}(\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c))$$

of K-closed subobjects in $[\mathcal{G}(\mathbb{P})^{\mathrm{op}}, \mathbf{Set}]$ of the presheaf $\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c)$.

Completions of fibred preorder sites

- The indexed functor $\eta_{\mathbb{P}}$ acts at an object $c \in \mathcal{C}$ as the functor

$$\eta_{\mathbb{P}}(c) : \mathbb{P}(c) \to L_{\mathcal{C}_{p_{\mathbb{P}}}}(c) = \mathrm{ClSub}^{K}_{[\mathcal{G}(\mathbb{P})^{\mathrm{op}}, \mathbf{Set}]}(\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c))$$

  sending any element $x \in \mathbb{P}(c)$ to the K-closure of the subfunctor of $\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c)$ sending any object (c', x') of $\mathcal{G}(\mathbb{P})$ to the subset $S_{(c',x')} \subseteq \mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}((c', x')), c) = \mathrm{Hom}_{\mathcal{C}}(c', c)$ consisting of the arrows $g : c' \to c$ such that $x' \leq \mathbb{P}(g)(x)$.

Remarks

- *This generalizes the completion of a preorder site $(\mathcal{C}, J)$ to the frame $\mathrm{Id}_J(\mathcal{C})$ of J-ideals on $\mathcal{C}$.*
- *It would be interesting to investigate the connection between this kind of completions and the exact completions for Lawvere doctrines and the tripos-to-topos construction.*
- *More generally, the notion of existential fibred site should illuminate the relationships between Grothendieck toposes as built from sites and elementary toposes as built from triposes.*

Towards relative geometric logic

The geometric approach to relative toposes which we have developed so far has a logical counterpart, which we may call relative geometric logic.

In its classical formulation, geometric logic does not have parameters embedded in its formalism; still, it is possible to introduce them without changing its degree of expressivity.

In a relative setting, parameters are fundamental if one wants to reason geometrically and use fibrational techniques. In fact, the semantics of stacks involves parameters in an essential way.

It turns out that the logical framework corresponding to relative toposes is a kind of fibrational, higher-order parametric logic in which it is possible to express a great number of higher-order constructions by using the parameters belonging to the base topos.

A theorem about 'elimination of parameters'

- The classical formulation of first-order logic by model theorists does not allow for a natural treatment of parameters, meaning sorts whose interpretation is fixed. So, for instance, while the notion of module over a variable ring can be very easily axiomatized, in order to formalize the notion of a R-module for a fixed ring R, one needs to introduce a function symbol f_r for each element r of R and axioms governing the relations between these symbols.
- To correct this asymmetry and lack of continuity between parameters and sorts, we have proved (with Raffaele Lamagna) a result showing that the syntax of ordinary geometric logic can be naturally extended through the introduction of constant sorts (i.e. sorts whose interpretation in any model is fixed rather than variable). Any theory in the extended signature can be shown to be Morita-equivalent, via a method of 'elimination of parameters', to a theory in the original signature.

Fibrations of theories

- From a geometrical point of view, any theory with a variable sort of 'type R' can be seen as a fibration of theories over the classifying topos for 'R-structures', each of which corresponding to a particular value of the parameter.
- Via this interpretation, turning a variable sort into a constant one corresponds to taking the fiber corresponding to that parameter, which is given by a fiber product of toposes.

For example, let $\mathbb{M}$ be the theory of (left) modules over a variable (commutative) ring, and let $\mathbb{R}$ be the theory of (commutative) rings. Then the classifying topos **Set**$[\mathbb{M}]$ is fibered over the classifying topos **Set**$[\mathbb{R}]$ of $\mathbb{R}$, and the fiber at a ring R internally to a topos $\mathcal{E}$, that is the theory $\mathbb{M}_R$ of R-modules (considered as a theory relative to $\mathcal{E}$), is given by the fibered product

$$\begin{tikzcd} \mathcal{E}[\mathbb{M}_R] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}[\mathbb{M}] \arrow[d] \\ \mathcal{E} \arrow[r, "\ulcorner R \urcorner"] & \mathbf{Set}[\mathbb{R}] \end{tikzcd}$$

where $\ulcorner R \urcorner$ is the morphism corresponding to the ring R via the universal property of the classifying topos **Set**$[\mathbb{R}]$.

Future developments

We plan to introduce a general stack semantics for 'automatically' transposing results from the absolute to the relative setting.

In particular, we intend to use this semantics to canonically obtain relative versions, formulated in the language of indexed categories and stacks, of the ordinary notions of limit and colimit, adjunction and Kan extension, separating set, filteredness and flatness, denseness conditions etc.

Among the main results that we expect to obtain there are:

- a relative version of Giraud's theorem (with Léo Bartoli);
- a theory of classifying toposes of (higher-order) relative geometric theories (with Raffaele Lamagna).

For further reading

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